Increasing stability in the continuation and inverse problems

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Introduction

The Cauchy Problem

$$(A + ck + k^2)u = f \text{ in } \Omega; \; u = u_0, \; \partial_\nu u = u_1 \text{ on } \Gamma \subset \partial\Omega. \qquad (1)$$

Here A is the linear partial differential operator of second order.
Applications: boundary control and inverse problems.
Uniqueness:
Holmgren-John (1900, 1950s): analytic coefficients; Carleman (1938): Carleman type estimates for non analytic coefficients; Calderon, Hörmander (1950-1970): systems, pseudo-convexity; Tataru (1995-2000).

Introduction

Carleman estimates need pseudo-convexity and imply Hölder type stability.

F. John (1960): when $\Omega = \{1 < |x| < R\}, A = \Delta$ the best stability estimate which is uniform with respect to the wave numbers k is of logarithmic type, i.e. bad for numerics. We will demonstrate that in a certain sense stability is always improving when k grows.

(Pseudo)convexity conditions

Let $m \times m$ matrix functions $\mathbf{B}_l, l = 1, ..., n$,
$\mathbf{C} = \mathbf{C}_1 k + \mathbf{C}_0 \in C^1(\bar{\Omega})$ and a positive $a \in C^2(\bar{\Omega})$. The Cauchy problem for the principally diagonal system

$$(\Delta + a^2 k^2 + \sum_{l=1}^{n} \mathbf{B}_l \partial_1 + \mathbf{C})\mathbf{u} = \mathbf{f} \text{ in } \Omega, \tag{2}$$

$$\mathbf{u} = \mathbf{u}_0, \partial_\nu \mathbf{u} = \mathbf{u}_1 \text{ on } \Gamma \subset \partial\Omega. \tag{3}$$

(Pseudo)convexity conditions

Let bounded $\Omega \subset \{0 < x_n < 1\}$ with Lipschitz $\partial\Omega$, $\bar{\Omega} \subset \{0 < x_n\}$ and $\Gamma = \partial\Omega \cap \{0 < x_n < 1\}$. Let $\Omega(d) = \Omega \cap \{x_n < 1 - d\}$.
$\|u\|_{(m)}(\Omega)$ is the norm in the Sobolev space $H^m(\Omega)$.
We let $F = \|\mathbf{f}\|(\Omega) + \|\mathbf{u}_0\|_{(1)}(\Gamma) + \|\mathbf{u}_1\|_{(0)}(\Gamma)$ and $F(k,d) = F + (k + d^{-1})\|\mathbf{u}_0\|_{(0)}(\Gamma)$. Constants $C = C(\Omega, \Gamma, a, \mathbf{B}_l, \mathbf{C}_1, \mathbf{C}_0)$.

(Pseudo)convexity conditions

Theorem

Let

$$0 < a + \nabla a \cdot x - \beta_n \partial_n a, \ \partial_n a \leq 0 \ on \ \overline{\Omega}. \quad (4)$$

Then there are $C, \lambda(d) \in (0,1)$ *such that*

$$\|\mathbf{u}\|_{(0)}(\Omega(d)) \leq C(F + k^{-1}(F^{\lambda_0} + d^{2\lambda_0} F^{\lambda_0}(k,d))M_1^{1-\lambda_0} +$$

$$k^{-1} d^{-\lambda_0} M_1^{1-\lambda(d)} F^{\lambda(d)}(k,d)) \quad (5)$$

for all $\mathbf{u}$ *solving* (2), (3). *Here* $\lambda_0 = \frac{1}{3}$ *and* $\|\mathbf{u}\|_{(1)})(\Omega) \leq M_1$.

Proof: I. (2007, 2009), Aralumallige, I. (2010)
Applicable to isotropic elasticity and Maxwell systems.

(Pseudo)convexity conditions

To prove (5) use stable extension of the Cauchy data and subtract it from **u**, then extend as zero onto $\Omega^* \setminus \Omega$, $\Omega^* = \{x : 0 < x_n < 1\}$. First let $a, \mathbf{B}_l, \mathbf{C}_1, \mathbf{C}_0$ depend only on x_n, and apply the Fourier transform **U** of **u** in $x' = (x_1, ..., x_{n-1})$ to obtain from (2)

$$\partial_n^2 \mathbf{U}(\xi',) + (a^2 k^2 - |\xi'|^2)\mathbf{U}(\xi',) + ... = \mathbf{F}(\xi',) \text{ on } (0, 1),$$

$$\mathbf{U}(\xi', 0) = \partial_n \mathbf{U}(\xi', 0) = 0.$$

(Pseudo)convexity conditions

(Scalarly) multiplying by $\partial_n \bar{\mathbf{U}} e^{-\tau x_n}$, using

$$\partial_n^2 \mathbf{U} \cdot \partial_n \bar{\mathbf{U}} + \partial_n^2 \bar{\mathbf{U}} \cdot \partial_n \mathbf{U} + (a^2 k^2 - |\xi'|^2)(\mathbf{U} \cdot \partial_n \bar{\mathbf{U}} + \bar{\mathbf{U}} \cdot \partial_n \mathbf{U}) =$$

$$|\partial_n \mathbf{U}|^2 + (a^2 k^2 - |\xi'|^2)\partial_n |\mathbf{U}|^2,$$

integrating by parts over $(0,1)$ and choosing large τ we obtain (Lipschitz) energy estimates

$$|\partial_n \mathbf{U}|^2(\xi, 1) ds + k^2 |\mathbf{U}|^2(\xi, 1) +$$

$$\int_0^1 |\partial_n \mathbf{U}|^2(\xi, s) ds + k^2 \int_0^1 |\mathbf{U}|^2(\xi, s) ds \leq C \int_0^1 |\mathbf{F}|^2(\xi, s) ds,$$

provided $\mathbf{U}(\xi', x_n) = 0$ if $|\xi'|^2 \geq (a^2 - \delta) k^2$ (low frequency part $\mathbf{u}_l$). ... denotes first order terms.

(Pseudo)convexity conditions

To handle high frequency part $\mathbf{u}_h$ we use

Theorem

Let the condition (4) *be satisfied.*
Then there are $C, \lambda_1(d) \in (0,1)$ *such that*

$$\|\mathbf{u}\|_{(1)}(\Omega(d)) \leq C(d^2 F(k,d) + d^{-2} M_1^{1-\lambda_1(d)} F^{\lambda_1(d)}(k,d)) \quad (6)$$

for all $\mathbf{u}$ *solving* (2), (3).

Proofs (I.(2009)) use a k independent Carleman type estimate obtained by using an associated with (2) wave equation.

(Pseudo)convexity conditions

To complete the proof of (5) use that

$$\|\mathbf{u}_h\|_{(0)} \leq Ck^{-1}\|\mathbf{u}_h\|_{(1)} \leq Ck^{-1}\|\mathbf{u}\|_{(1)}$$

and combine Lipschitz stability for $\mathbf{u}_l$ with (6). To use energy estimates for x'-independent coefficients: freeze coefficients in x' and use partition of the unity.

No convexity conditions

Let $\Omega = \{x : 1 < |x| < R\}$ in $\mathbb{R}^2$ and $\Gamma = \{|x| = 1\}$. In polar coordinates (ϕ, r) for a function

$$u(\phi, r) = \sum_{n=0}^{\infty} (u_{n1}(r) cosn\varphi + u_{n2}(r) sinn\phi)$$

we let

$$u^N(\phi, r) = \sum_{n=0}^{N} (u_{n1}(r) cosn\phi + u_{n2}(r) sinn\phi)$$

No convexity conditions

Lemma

Let $N = \frac{k}{\sqrt{2}}$ *and* $\Gamma_1 = \{x : |x| = R\}$*. Then*

$$R^{-2}\|\partial_\nu u^N\|^2_{(0)}(\Gamma_R) + \frac{1}{2}R^{-2}\|\partial_\phi u^N\|^2_{(0)}(\Gamma_R) + \frac{1}{4}R^{-2}k^2\|u^N\|^2_{(0)}(\Gamma_R) \leq$$

$$\|u_1^N\|^2_{(0)}(\Gamma) + k^2\|u_0^N\|^2_{(0)}(\Gamma).$$

To prove: use energy estimates (multiply by $r^{-4}\partial_r u_{nj}$ and integrate by parts over Ω) for the Bessel's equations

$$r\partial_r(r\partial_r u_{nj}) + (k^2 r^2 - n^2)u_{nj} = 0.$$

Lemma and next numerical examples from I.,Kindermann(2010)

No convexity conditions

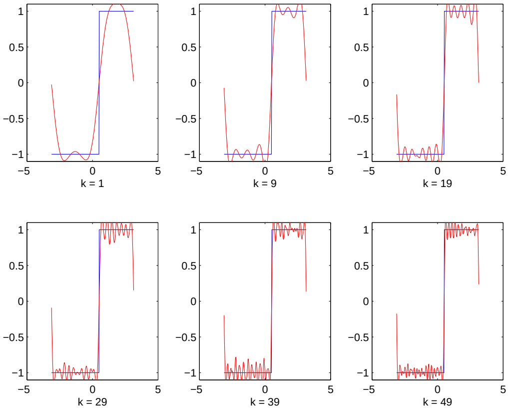

Figure: Recovery of the Heavyside function on a circle for increasing k.

No convexity conditions

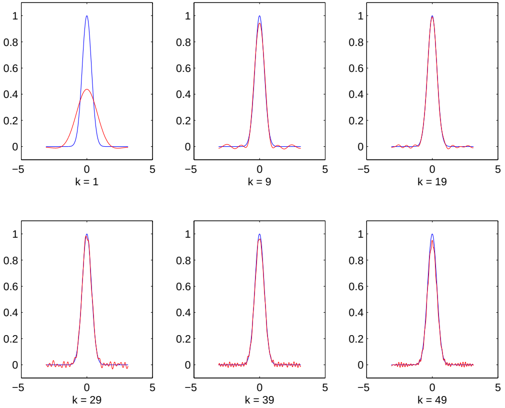

Figure: Recovery of Gaussian distribution.

No convexity conditions

Let $\Omega \subset \mathbb{R}^{n-1} \times (0,1)$, $\Gamma = \partial\Omega \cap \{x_n = 0\}$, $\Gamma_1 = \partial\Omega \cap \{x_n = 1\}$. Let V be a neighborhood of $\partial\Omega \cap (\mathbb{R}^{n-1} \times [0,1])$ and $\omega = \Omega \cap V$. Let $\chi \in C^\infty$, $\chi = 0$ outside Ω, $\chi = 1$ on $\Omega \setminus V$. We define $v = \chi u$. We consider elliptic $Au = \sum_{j,m=1}^{n} a_{jm}\partial_j\partial_m u + ... + cku$ with C^1-coefficients. We have $\sum_{j,m=1}^{n-1} a_{jm}\xi_j\xi_m \leq E^2|\xi|^2$. Let

$$v_l(x) = \mathcal{F}^{-1}\chi(E)\mathcal{F}v(x)$$

where $\mathcal{F}$ is the Fourier transform in $x' = (x_1, ..., x_{n-1})$ and $\chi(E)(\xi') = 1$ if $|\xi'| < (1-\delta)\frac{k}{E}$ and $\chi(E) = 0$ if $|\xi'| > (1-\delta)\frac{k}{E}$.

No convexity conditions

Theorem

Let u solve (1). Let $\theta > 0$.
There are a monotone family of closed subspaces $H_{(2)}(\Omega; k)$ of $H_{(2)}(\Omega)$ with $\cup_k H_{(2)}(\Omega; k) = H_{(2)}(\Omega)$, linear continuous operators P_k from $H_{(2)}(\Omega)$ onto $H_{(2)}(\Omega; k)$ with $P_k u_l = u_l$ for $u_l \in H_{(2)}(\Omega; k)$, and a constants $C, C(\theta)$ such that

$$\|u\|_{(1)}(\Gamma_1 \setminus V) + \|\nabla u\|_{(0)}(\Gamma_1 \setminus V) + \|u\|_{(1)}(\Omega) \leq$$

$$CF + C(\theta) k^{-\frac{1}{2}+\theta} ||u - u_l||_{(2)}(\Omega)) \tag{7}$$

where $u_l = P_k u$,
$F = \|f\|_{(0)}(\Omega) + \|u_0\|_{(1)}(\Gamma) + \|u_1\|_{(0)}(\Gamma) + \|u\|_{(1)}(\omega)$.

No convexity conditions

Since conditions are invariant with respect to C^2-diffeomorphisms, Ω can be replaced by its image under such diffeomorphism
A proof of (7) follows from energy integrals: multiply

$$a_{nn}\partial_n^2 v + 2\sum_{j=1}^{n-1} a_{jn}\partial_j\partial_n v + \sum_{j,m=1}^{n-1} a_{jm}\partial_j\partial_m v + ... + k^2 v = \chi f + A_1 u$$

by $\partial_n v e^{-\tau x_n}$ and integrate by parts over Ω) to yield

$$\int_{\mathbb{R}^{n-1}} a_{nn}(\partial_n v)^2(,1)e^{-\tau} + k^2\int_{\mathbb{R}^{n-1}} v^2(,1)e^{-\tau} -$$

$$\int_{\mathbb{R}^{n-1}} \sum_{j,m=1}^{n-1} a_{jm}\partial_j v\partial_m v(,1)e^{-\tau} + ... \leq (data)$$

No convexity conditions

To bound the last term, let $v = v_l + v_h$ and use that

$$-\int_{\mathbb{R}^{n-1}} \sum_{j,m=1}^{n-1} a_{jm}\partial_j v_l \partial_m v_l \geq -(1-\delta)k^2 \int_{\mathbb{R}^{n-1}} v_l^2.$$

The terms containing v_h are bounded by $Ck^{-\frac{1}{2}+\theta}\|u\|_{(2)}$ by using Extension and Trace theorems and basic Fourier analysis.

Let Ω be in the unit ball of $\mathbb{R}^3$. Assume $c \in L_\infty(\Omega)$ The Dirichlet problem

$$-\Delta u - k^2 u + cu = 0 \text{ in } \Omega,\ u = g \text{ on } \partial\Omega,$$

generates the Dirichlet-to-Neumann map $\Lambda_c g = \partial_\nu u$ on $\partial\Omega$.
Uniqueness of c from Λ_c: Sylvester and Uhlmann (1987).
Logarithmic stability (Alessandrini (1987)) is optimal (Mandache (2000)).
Λ_c is a continuous linear operator from $H^{\frac{1}{2}}(\Gamma)$ into $H^{-\frac{1}{2}}(\Gamma)$ with the norm $||\Lambda_c||$. We assume that c is zero near $\partial\Omega$. C_0 generic constants (not depending on c, k, or Ω). Let $\varepsilon = \|\Lambda_{c_2} - \Lambda_{c_1}\|, E = -log\varepsilon \geq 2$. Let

$$||c_j||_\infty(\Omega) \leq M,\ ||c_j||_{1,\infty}(\Omega) \leq M_1,\ j = 1, 2.$$

Theorem

There are C_0, C_Ω such that if

$$k \leq \frac{E^2}{2} - \frac{E}{4},\ C_0^2 M < \sqrt{\frac{E^2}{2} - \frac{E}{4} - k} + 2k^2 + 4,$$

then

$$||c_2 - c_1||_2(\Omega) \leq C_0 M^3 (E+k)^{-\frac{1}{4}} + \frac{M_1}{\sqrt{E+k}} +$$

$$C_\Omega E^2 (E^2 + M^2) \varepsilon^{1-\frac{1}{\sqrt{2}}}.$$

If $E \leq k,\ C_0^2 M^2 < k^2 + 2$, then

$$||c_2 - c_1||_2(\Omega) \leq \frac{C_0 + M_1}{\sqrt{k+E+1}} + C_\Omega (k + M^2) k \varepsilon.$$

Proofs
I.(2011): complex and real geometrical optics.
I., Nagayasu, Uhlmann, and Wang (2013): under additional smoothness of c one can use only complex geometrical optics and get better stability.
Isaev, R. Novikov (2012): use of scattering solutions.

Conclusion

Anisotropic (elasticity and Maxwell) systems can also be handled without convexity conditions.

Further research:

1) numerical evidence of the increasing stability for more complicated geometries and for systems;

2) weaker constraint in (7);

3) without (pseudo)convexity condition (like(4)) replace (6) by a logarithmic type estimate;

4) generalizatons to parabolic and hyperbolic equations (in time-like Cauchy problem);

5) increasing stability for a in $\Delta + k^2 a$ from all boundary measurements; numerical evidence (Natterer, Wubbeling (1995)), first results (Nagayasu, Uhlmann, Wang (2012)).

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Papers are on the web site http://www.math.wichita.edu/ isakov/