Determination of heat transfer coefficients

D. Lesnic
Department of Applied Mathematics, University of Leeds, UK

Heat Transfer Law

$$k\frac{\partial T}{\partial n} = \sigma(T^{ambient} - T)^{\beta} + B, \quad \text{on the boundary}$$

where
$T =$ temperature
$T^{ambient} =$ ambient temperature
$k =$ thermal conductivity
$n =$ outward unit normal to the boundary
$B =$ additional heat flux
$\sigma =$ heat transfer coefficient (may be space-, time-, or temperature-dependent)
$\beta = 1$ for convection; $\beta = 4$ for radiation.

Outline

Mathematical formulation

- Space-dependent heat transfer coefficient
- Time-dependent heat transfer coefficient
- Temperature-dependent heat transfer coefficient

Boundary element method (BEM)

Numerical Results and Discussion

Conclusions

1. Space-dependent Heat Transfer Coefficient

Consider the inverse problem which requires finding the temperature $T \in C^{2,1}(Q)$ and the space-dependent heat transfer coefficient $\sigma \in C(\partial\Omega)$, $\sigma \geq 0$, satisfying the heat equation

$$\frac{\partial T}{\partial t}(x,t) = \nabla^2 T(x,t), \quad (x,t) = \Omega \times (0, t_f] =: Q,$$

subject to the initial condition

$$T(x,0) = T_0(x), \quad x \in \Omega,$$

the Robin boundary condition

$$\frac{\partial T}{\partial n}(x,t) + \sigma(x)T(x,t) = B(x,t), \quad (x,t) \in \partial\Omega \times (0, t_f),$$

and the instant temperature observation at the fixed time $t^0 \in (0, t_f)$:

$$T(x, t^0) = \chi(x), \quad x \in \partial\Omega$$

or, the additional integral time-average temperature observation

$$\int_0^{t_f} \omega(t) T(x,t) dt = \chi(x), \quad x \in \partial\Omega,$$

where $\omega \in L_1(0, t_f)$ is given.

Boundary Element Method (BEM)

Using the BEM we reduce the inverse problem to nonlinear boundary integral equations for the boundary temperature and the heat transfer coefficient:

$$\frac{1}{2}T(x,t) = \int_\Omega G(x,t;y,0)T_0(y)d\Omega(y)$$
$$+ \int_0^t \int_{\partial\Omega} B(\xi,\tau)G(x,t;\xi,\tau)dS(\xi)d\tau$$
$$-\int_0^t \int_{\partial\Omega} T(\xi,\tau)\left[\frac{\partial G}{\partial n(\xi)}(x,t;\xi,\tau) + \sigma(\xi)G(x,t;\xi,\tau)\right]dS(\xi)d\tau,$$
$$(x,t) \in \partial\Omega \times (0,t_f),$$

where

$$G(x,t;\xi,\tau) = \frac{H(t-\tau)}{[4\pi(t-\tau)]^{n/2}} \exp\left(-\frac{\|x-\xi\|^2}{4(t-\tau)}\right)$$

is the fundamental solution of the heat equation and H is the Heaviside function.

Numerical Example

Find the temperature $T(x,t)$ $(= x^2 + 2t)$ and the space-dependent heat transfer coefficient(s) $0 \leq \sigma_0$ $(= 1)$, $0 \leq \sigma_1$ $(= 1)$ solving the problem

$$\frac{\partial T}{\partial t}(x,t) = \frac{\partial^2 T}{\partial x^2}(x,t), \quad (x,t) = (0,1) \times (0, t_f = 1],$$
$$T(x,0) = x^2, \quad x \in [0,1],$$
$$-\frac{\partial T}{\partial x}(0,t) + \sigma_0 T(0,t) = 2t, \quad t \in (0,1),$$
$$\frac{\partial T}{\partial x}(1,t) + \sigma_1 T(1,t) = 2t + 3, \quad t \in (0,1),$$

and the additional 1% noisy measurement conditions

$$T(0,t^0) = 2t^0 * 1.01, \quad T(1,t^0) = (1 + 2t^0) * 1.01,$$

or

$$\int_0^{t_f} T(0,t)dt = 1 * 1.01, \quad \int_0^{t_f} T(1,t)dt = 2 * 1.01.$$

Using the BEM with $N = N_0 = 40$ elements, in the latter case we have obtained: $\sigma_0 = 0.9875$ and $\sigma_1 = 0.9777$. In the former case see the figure on the next slide.

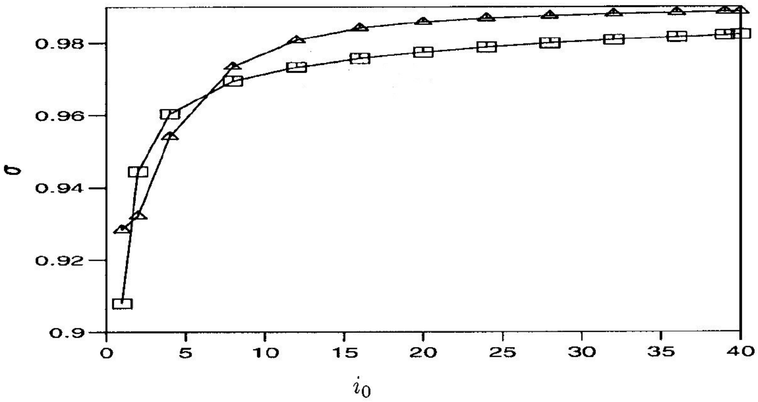

Figure 6. The constants σ_0 ($\triangle$) and σ_1 ($\square$) for Problem I, as a function of $i_0 = 1, \ldots N = 40$, when $(N_0, N) = (40, 40)$ (1% noise).

2. Time-dependent Heat Transfer Coefficient

Consider the inverse problem which requires finding the temperature $T \in C^{2,1}(Q)$ and the time-dependent heat transfer coefficient $\sigma \in C([0, t_f])$ satisfying the heat equation

$$\frac{\partial T}{\partial t}(x,t) = \nabla^2 T(x,t), \quad (x,t) = \Omega \times (0, t_f] =: Q,$$

subject to the initial condition

$$T(x,0) = T_0(x), \quad x \in \Omega,$$

the Robin boundary condition

$$\frac{\partial T}{\partial n}(x,t) + \sigma(t) T(x,t) = B(x,t), \quad (x,t) \in \partial\Omega \times (0, t_f),$$

and the temperature observation at the fixed point $x_0 \in \partial\Omega$:

$$T(x_0, t) = \overline{\chi}(t), \quad t \in [0, t_f]$$

or, the additional boundary integral temperature observation

$$\int_{\partial\Omega} \nu(x) T(x,t) dS(x) = \overline{\chi}(t), \quad t \in [0, t_f],$$

where $\nu \in L_1(\partial\Omega)$ is given.

Numerical Example

Find the temperature $T(x,t)$ $(= x^2 + 2t + 1)$ and the time-dependent heat transfer coefficient $\sigma(t)$ $(= t)$, solving the problem

$$\frac{\partial T}{\partial t}(x,t) = \frac{\partial^2 T}{\partial x^2}(x,t), \quad (x,t) = (0,1) \times (0, t_f = 1],$$

$$T(x,0) = x^2, \quad x \in [0,1],$$

$$-\frac{\partial T}{\partial x}(0,t) + \sigma(t)T(0,t) = 2t^2 + t, \quad t \in (0,1),$$

$$\frac{\partial T}{\partial x}(1,t) + \sigma(t)T(1,t) = 2(t^2 + t + 1), \quad t \in (0,1),$$

and the additional $\rho\%$ noisy measurement condition

$$T(0,t) = 2t + 1 + \epsilon, \quad t \in (0,1),$$

where ρ denotes the percentage of noise and ϵ are random variables taken from a Gaussian normal distribution with zero mean and stadard deviation $3\rho\%$.

Using the BEM with $N = N_0 = 40$ elements and various amounts of noise $\rho\% \in \{1, 3, 5\}\%$ we obtain the figure on the next slide.

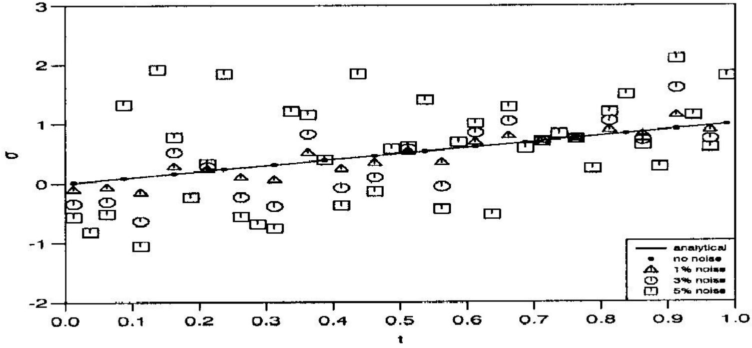

Figure 11. The analytical and numerical heat transfer coefficients $\sigma(t)$ for Problem II, as functions of time t, for various amounts of noise.

2'. Time-dependent Heat Transfer Coefficient

Consider the inverse problem which requires finding the pair solution $(T(x,t),\sigma(t)) =$ (temperature,heat transfer coefficient) satisfying the heat equation

$$\frac{\partial T}{\partial t}(x,t) = \nabla^2 T(x,t), \quad (x,t) = \Omega \times (0,t_f] =: Q,$$

subject to the initial condition

$$T(x,0) = T_0(x), \quad x \in \Omega,$$

the Robin boundary conditions

$$\frac{\partial T}{\partial n}(x,t) + \sigma(t) g(T(x,t)) = B(x,t), \quad (x,t) \in \partial\Omega \times (0,t_f),$$

and the additional boundary integral (non-local) observation

$$\int_{\partial\Omega} \Phi(T(x,t)) dS(x) = E(t), \quad t \in [0,t_f],$$

where $\Phi(T) = \int^T g(s)ds$ denotes a primitive (anti-derivative) of g.

Remarks:

- Of physical interest is the linear convection case $g(T) = T$, and the nonlinear radiative case $g(T) = T^3|T|$.

- Multiplying with T the heat equation and integrating over Ω results in

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega T^2(x,t)d\Omega\right) + \int_\Omega |\nabla T|^2 d\Omega = \int_{\partial\Omega} T\ h\ dS$$
$$-\sigma(t)\int_{\partial\Omega} g(T)T\ dS$$

and one could recognise the last term as an 'energy' term $(\alpha+1)\sigma(t)E(t)$ for the nonlinearity $g(T) = T^\alpha$.

Let us now consider the weak solutions T and σ of the inverse problem defined in the following spaces of functions:

$$T \in C([0, t_f], L_2(\Omega)) \cap L_2((0, t_f), H^1(\Omega))$$
$$\text{with } \partial_t T \in L_2((0, t_f), L_2(\Omega)).$$

$\sigma \geq 0$ and $\sigma \in C^1[0, t_f]$ with σ'/σ bounded.

We also require that the input data be such that:

$$T_0 \in H^2(\Omega), \quad B, B_t \in L_2((0, t_f), L_2(\partial\Omega)),$$
$$g' \geq 0, \quad g(0) = 0, \quad |g(s)| \leq C(|s|^\alpha + 1)$$

for some non-negative constants C_0, C and α.

Definition. *For a given* $\sigma \in L_2(0, t_f)$, $\sigma \geq 0$, *a function* $T_\sigma \in L_2((0, t_f), H^1(\Omega))$ *with* $\partial_t T \in L_2((0, t_f), L_2(\Omega))$ *is called a weak solution to the direct problem if* $T_\sigma(x, 0) = T_0(x)$ *and*

$$(\partial_t T_\sigma, \phi) + (\partial_x T_\sigma, \partial_x \phi) + \sigma(g(T_\sigma), \phi)_{\partial\Omega} = (B, \phi)_{\partial\Omega},$$
$$\forall \phi \in H^1(\Omega), \text{ a.e. in } (0, t_f).$$

Theorem. (unique solvability of the direct problem)
There exists a unique weak solution to the direct problem.

Existence and Uniqueness Theorem. (Slodicka and Lesnic (2010))
Assume that a compatibility condition at $t = 0$ *holds and that* $E'(t) \geq \delta_0 > 0$, $|E''(t)| \leq C_0$, $\forall t \in [0, t_f]$ *and that*

$$0 < E(t) \leq \int_{\partial\Omega} \Phi(T^0(x,t)) dS(x), \quad \forall t \in [0, t_f],$$

where T^0 *is the unique weak solution of the direct problem with* $\sigma = 0$*. Then there exists a unique solution to the inverse problem.*

The continuous dependence of the solution on the input energy data $E(t)$ can (probably) be established under the additional assumption that σ is bounded. This is an usual additional source condition which when imposed onto the solution of some ill-posed problems restore its stability with respect to noise added into the input data.

We employ the BEM

$$\frac{1}{2}T(x,t) = \int_{\Omega} G(x,t;y,0)T_0(y)d\Omega(y)$$
$$+\int_0^t \int_{\partial\Omega} [B(\xi,\tau) - g(T(\xi,\tau))\sigma(\tau)]G(x,t;\xi,\tau)dS(\xi)d\tau$$
$$-\int_0^t \int_{\partial\Omega} T(\xi,\tau)\frac{\partial G}{\partial n(\xi)}(x,t;\xi,\tau)dS(\xi)d\tau,$$
$$\forall(x,t)\partial\Omega \times (0,t_f],$$

and

$$\int_{\partial\Omega} \Phi(T(\xi,t))dS(\xi) = E(t), \quad \forall t \in (0,t_f].$$

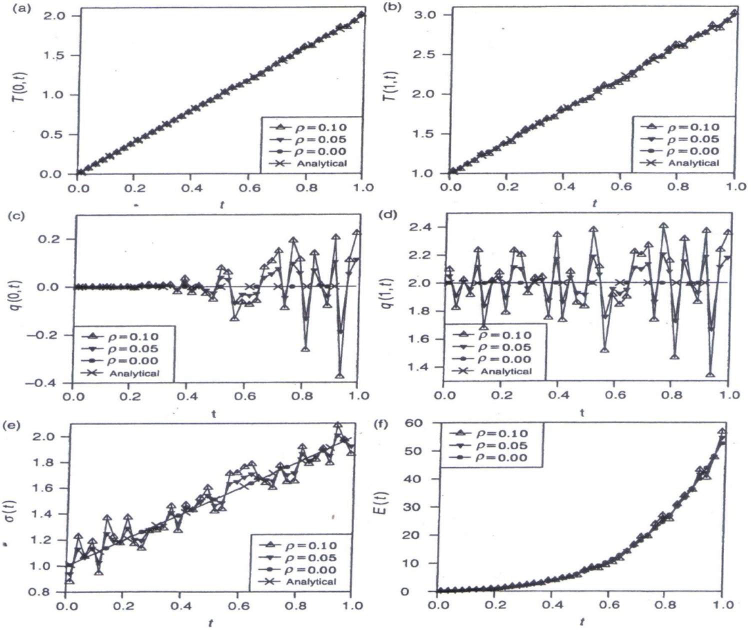

Figure 2. The analytical and numerical boundary temperatures (a) $T(0, t)$ and (b) $T(1, t)$, the heat

3. Temperature-dependent Heat Transfer Coefficient

Consider the inverse problem of finding the temperature $T \in C^{3,3/2}(\overline{Q})$ and the space-dependent heat transfer coefficient $\sigma \in C^1([\theta_1, \theta_2])$, where $\theta_1 = min_{\overline{Q}} u(x,t)$ and $\theta_2 = max_{\overline{Q}} u(x,t)$ are assumed known *a priori* and satisfy $\theta_1 \theta_2 > 0$. We also assume

$$\overline{\chi}(0) \leq u(x,t) \leq \overline{\chi}(t), \quad (x,t) \in \partial\Omega \times [0, t_f].$$

In addition, the pair solution $(T, \sigma(T))$ satisfies the heat equation

$$\frac{\partial T}{\partial t}(x,t) = \nabla^2 T(x,t), \quad (x,t) = \Omega \times (0, t_f] =: Q,$$

subject to the initial condition

$$T(x,0) = T_0(x), \quad x \in \Omega,$$

the Robin boundary condition

$$\frac{\partial T}{\partial n}(x,t) + \sigma(T(x,t))T(x,t) = B(x,t), \quad (x,t) \in \partial\Omega \times (0, t_f),$$

and the temperature observation at the fixed point $x_0 \in \partial\Omega$:

$$T(x_0, t) = \overline{\chi}(t), \quad t \in [0, t_f].$$

Uniqueness Theorem. (Rundell and Yin (1990))
If $B \in C^{2,2}(\partial\Omega \times [0, t_f])$, and $\overline{\chi} \in C^2([0, t_f])$ is strictly increasing, then the solution is unique.

Further, in the one-dimensional case we seek $T \in C^{2,1}(Q)$ and

$$\sigma \in \Sigma_{adm} := \{\sigma \in C^{0+1}([\theta_1, \theta_2]) | 0 < m_1 \leq \sigma(T) \leq M_1 < \infty\},$$

where $\theta_1 = min\{0, inf_{x \in (0,1)} T_0(x)\}$ and
$\theta_2 = max\{0, max_{x \in (0,1)} T_0(x)\}$.

Existence and Uniqueness Theorem. (Pilant and Rundell (1989))
In the one-dimensional case, if $T_0 \in C^{2+1/2}([0, 1])$, $B = 0$, and $\overline{\chi} \in C^{1+1/2}([0, t_f])$ is strictly monotone and $\overline{\chi}(0) = T_0(0) = T_0(1)$, then the inverse problem has a unique solution.

Boundary Element Method (BEM)

Using the BEM we reduce the inverse problem to nonlinear boundary integral equations for the boundary temperature and the heat transfer coefficient:

$$\frac{1}{2}T(x,t) = \int_{\Omega} G(x,t;y,0)T_0(y)d\Omega(y) + \int_0^t \int_{\partial\Omega} B(\xi,\tau)G(x,t;\xi,\tau)dS(\xi)d\tau$$
$$- \int_0^t \int_{\partial\Omega} T(\xi,\tau)\left[\frac{\partial G}{\partial n(\xi)}(x,t;\xi,\tau) + \sigma(T(\xi,\tau))G(x,t;\xi,\tau)\right] dS(\xi)d\tau,$$
$$(x,t) \in \partial\Omega \times (0,t_f).$$

In one-dimension, with the temperature measurement taken at the boundary point $x_0 = 0$ we obtain a coupled system of two nonlinear boundary integral equations in two unknowns, namely $T(1,t)$ and $\sigma(T(1,t))$:

$$\frac{1}{2}\overline{\chi}(t) = \int_0^1 G(0,t;y,0)T_0(y)dy$$
$$+\int_0^t \overline{\chi}(t)\left[G(0,t;0,\tau)\sigma(\overline{\chi}(t)) + \frac{\partial G}{\partial \xi}(0,t;0,\tau)\right]d\tau$$
$$+\int_0^t T(1,t)\left[G(0,t;1,\tau)\sigma(T(1,t)) - \frac{\partial G}{\partial \xi}(0,t;1,\tau)\right]d\tau, \quad t \in (0,t_f),$$

$$\frac{1}{2}T(1,t) = \int_0^1 G(1,t;y,0)T_0(y)dy$$
$$+\int_0^t \overline{\chi}(t)\left[G(1,t;0,\tau)\sigma(\overline{\chi}(t)) + \frac{\partial G}{\partial \xi}(1,t;0,\tau)\right]d\tau$$
$$+\int_0^t T(1,t)\left[G(1,t;1,\tau)\sigma(T(1,t)) - \frac{\partial G}{\partial \xi}(1,t;1,\tau)\right]d\tau, \quad t \in (0,t_f).$$

Using a constant BEM approximation with N boundary elements and N_0 cells, we obtain a system of $2N$ nonlinear equations

$$A_\sigma(\underline{T_1}) = \underline{b},$$

where $\underline{T}_1$ contains $T(1,t)$, $\underline{b}$ contains T_0 and B, and A_σ is a nonlinear operator depending on σ. Assuming that $\overline{\chi}$ is strictly increasing, let

$$q_k := \overline{\chi}(0) + k(\overline{\chi}(t_f) - \overline{\chi}(0))/K, \quad k = \overline{0,K}$$

denote a uniform discretisation of the the interval $[\overline{\chi}(0), \overline{\chi}(t_f)]$ into K equal sub-intervals. Then we seek a piecewise constant function $\sigma(T)T =: f : [q_0, q_K] \to \mathbb{R}$ defined by

$$f(T) = \begin{cases} a_1, & T \in [q_0, q_1) \\ a_2, & T \in [q_1, q_2) \\ \vdots & \vdots \\ a_K, & T \in [q_{K-1}, q_K) \end{cases}$$

where the unknown coefficients $\underline{a} = (a_k)_{k=\overline{1,K}}$ are yet to be determined.

Assuming $T(1,t;f) \in [\overline{\chi}(0), \overline{\chi}(t)]$, $\forall t \in [0, t_f]$, we have

$$f(\overline{\chi}(\tilde{t}_l)) = a_{\phi(l)}, \quad f(T(1,\tilde{t}_l)) = a_{\psi(l)}, \quad l = \overline{1,l},$$

where $\tilde{t}_l$ are the boundary element nodes, and for each $l \in \{1, ..., N\}$, $\phi(l)$ is the unique number in the set $\{1, ..., K\}$ such that $\overline{\chi}(\tilde{t}_l) \in [q_{\phi(l)-1}, q_{\phi(l)})$, and $\psi(l)$ is the unique number in the set $\{1, ..., K\}$ such that $T(1, \tilde{t}_l) \in [q_{\psi(l)-1}, q_{\psi(l)})$.

We then minimize (using the NAG routine E04FCF) the nonlinear Tikhonov functional

$$S : \mathbb{R}^K \times \mathbb{R}^N \to \mathbb{R}_+, \quad S(\underline{a}, \underline{T}_1) := \|A_\sigma(\underline{T}_1) - \underline{b}\|^2 + \kappa\|\underline{a}\|^2,$$

where $\kappa > 0$ is a regularization parameter to be prescribed.

Numerical Examples

The BEM is applied with $(N, N_0) = (40, 40)$ to generate the forward operator $A_\sigma(\underline{T}_1)$. The piecewise constant parametrisation of $f(T) = \sigma(T)T$ is sought with $K = 10$.

The analytical temperature to be retrieved

$$T(x,t) = x^2 - x + 1 + 2t, \quad (x,t) \in [0,1] \times [0,1],$$

generates the initial temperature

$$T(x,0) = T_0(x) = x^2 - x + 1, \quad x \in [0,1],$$

and the boundary temperature measurement

$$T(0,t) = \overline{\chi}(t) = 1 + 2t, \quad t \in [0,1].$$

Remark that $\overline{\chi}$ is strictly increasing and that $T(1,t) = 1 + 2t \in [\overline{\chi}(0) = 1, \overline{\chi}(t) = 1 + 2t]$, $\forall t \in [0,1]$, such that the unique solvability of the inverse problem is ensured.

Numerical results are presented next for $f(T) \in \{1, T^4\}$ which corresponds to a heat transfer coefficient $\sigma(T) \in \{T^{-1}, T^3\}$.

Example 1. $f(T) = 1$. Initial guess $(\underline{a}^0, \underline{T}_1^0) = (\underline{3}, \underline{3})$.

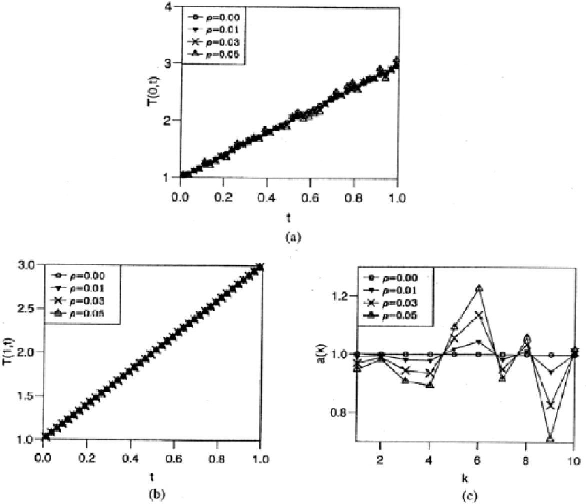

Figure 1: (a) The analytical boundary temperature $T(0,t)$, (b) the numerical boundary temperature $T(1,t)$, as functions of time t, and (c) the numerical vector $\mathbf{a} = (a_k)_{k=\overline{1,10}}$, when the amount of noise in (4.4) is: (∘)

Example 2. $f(T) = T^4$. Initial guess $(\underline{a}^0, \underline{T}_1^0) = (\underline{50}, \underline{3})$ and the Neumann conditions (1) and (2) modified as

$$\frac{\partial T}{\partial n}(x,t) = f(T(x,t)) + 1 - (1+2t)^4, \quad (x,t) \in \{0,1\} \times (0,1].$$

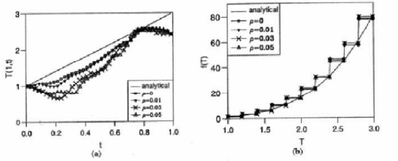

Figure 2: The analytical and numerical approximations of (a) the boundary temperature $T(1,t)$ and (b) the function $f(T)$, when $\rho \in \{0,1,3,5\}\%$ and $\kappa = 0$.

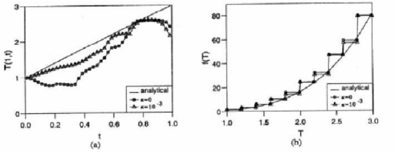

Figure 3: The analytical and numerical approximations of (a) the boundary temperature $T(1,t)$ and (b) the function $f(T)$, when $\rho = 5\%$, $\kappa = 0$ and 10^{-3}.

4. Conclusions

• Reconstruction of heat transfer coefficient which may be space-, time-, or temperature-dependent has been addressed.

• The existence and uniqueness of solution has been discussed in both strong and weak senses. Furthermore, a numerical method based on the boundary element method (BEM) (combined with the Tikhonov reqularization method where necessary) has been devised in order to obtain stable and accurate numerical solutions.

• Future work will investigate iterative regularizations and higher-dimensional numerical reconstructions.