Geometry of optimal decomposition for the L- functional and duality in convex analysis

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ROF model

ROF model

In 1992 Rudin, Osher and Fatemi suggested a denoising model which has made great success. Let $D = [a, b] \times [c, d]$ be a rectangular domain in $\mathbb{R}^2$. Suppose that initial image $f_* \in BV$ and we observe

$$f_{\mathrm{ob}} = f_* + \eta,$$

where $\eta \in L^2(D)$ corresponds to noise. In order to reconstruct approximately initial image f_*, ROF suggested to consider

$$L_{2,1}\left(t, f_{\mathrm{ob}}, L^2(D), BV(D)\right) = \inf_{g \in BV}\left(\frac{1}{2}\left\|f_{\mathrm{ob}} - g\right\|_{L^2}^2 + t\left\|g\right\|_{BV}\right),$$

and to take as approximation to f_* the function f_t which minimizes this functional, i.e.,

ROF model

$$L_{2,1}\left(t, f_{\rm ob}, L^2(D), BV(D)\right) = \frac{1}{2}\left\|f_{\rm ob} - f_t\right\|_{L^2}^2 + t\left\|f_t\right\|_{BV},$$

where (for a function f of class C^1)

$$\|f\|_{BV} = \iint\limits_D \left(\left|\frac{\partial f}{\partial x}(x,y)\right| + \left|\frac{\partial f}{\partial y}(x,y)\right|\right) dxdy.$$

Optimal decomposition

The expression

$$f_{\rm ob} = (f_{\rm ob} - f_t) + f_t,$$

is called *optimal decomposition* of $L_{2,1}\left(t, f_{\rm ob}, L^2(D), BV(D)\right)$ corresponding to $f_{\rm ob}$.

ROF model

In 2002, Yves Meyer obtained a mathematical characterization of this optimal decomposition for this couple $\left(L^2(D), BV(D)\right)$ by using duality. [*Yves Meyer, Oscillating Patterns in Image Processing and Nonlinear Evolution Equations, 2002*]

The problem

Let (X_0, X_1) be a compatible Banach couple. i.e., X_0 and X_1 are Banach spaces such that X_0 and X_1 are linearly and continuously embedded in some Banach space $\mathcal{X}$. Let $x \in X_0 + X_1$, let $1 < p < +\infty$ and $t > 0$. We consider the L- functional

$$L_{p,1}(t, x; X_0, X_1) = \inf_{x = x_0 + x_1} \left(\frac{1}{p} \|x_0\|_{X_0}^p + t \|x_1\|_{X_1} \right),$$

We give a characterization of *optimal decomposition* for the L-functional. i.e., $x = x_{0,\mathrm{opt}} + x_{1,\mathrm{opt}}$ such that

$$L_{p,1}(t, x; X_0, X_1) = \frac{1}{p} \|x_{0,\mathrm{opt}}\|_{X_0}^p + t \|x_{1,\mathrm{opt}}\|_{X_1}$$

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For ROF model,

$$p = 2,\ X_0 = L^2(D),\ X_1 = BV(D)$$

Dual characterization of optimal decomposition

Let (X_0, X_1) be a regular couple ($X_0 \cap X_1$ is dense in both X_0 and X_1). Then it is a known fact from interpolation theory that (X_0^*, X_1^*) also form a Banach couple and $(X_0 \cap X_1)^* = X_0^* + X_1^*$. The dual spaces are defined by the norm

$$\|y\|_{X_j^*} = \sup\left\{\langle y, x\rangle : \ x \in X_j, \ \|x\|_{X_j} \leq 1\right\}, \ j = 0, 1.$$

The spaces $X_0 + X_1$ and $X_0 \cap X_1$ are Banach spaces with respect to the following norms

$$\|x\|_{X_0+X_1} = \inf_{x=x_0+x_1} \left\{\|x_0\|_{X_0} + \|x_1\|_{X_1}\right\},$$

where the infimum extends over all representations $x = x_0 + x_1$ of x with x_0 in X_0 and x_1 in X_1, and

$$\|x\|_{X_0\cap X_1} = \max\left\{\|x\|_{X_0}, \|x\|_{X_1}\right\}.$$

Dual characterization of optimal decomposition

Theorem (Main Theorem)

Let $1 < p < +\infty$. The decomposition $x = x_{0,opt} + x_{1,opt}$ is optimal for $L_{p,1}(t, x; X_0, X_1)$ if and only if $\exists y_* \in X_0^* \cap X_1^*$ such that $\|y_*\|_{X_1^*} \leq t$ and

$$\begin{cases} \frac{1}{p}\|x_{0,opt}\|_{X_0}^p = \langle y_*, x_{0,opt}\rangle - \frac{1}{p'}\|y_*\|_{X_0^*}^{p'}; \\ t\|x_{1,opt}\|_{X_1} = \langle y_*, x_{1,opt}\rangle, \end{cases}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Couple (ℓ^p, X)

Consider a particular but important case of couple (ℓ^p, X) on $\mathbb{R}^n$

$$L_{p,1}(t, x; \ell_p, X) = \inf_{x = x_0 + x_1} \left(\frac{1}{p} \|x_0\|_{\ell_p}^p + t \|x_1\|_{\mathbf{X}} \right),$$

where $1 < p < +\infty$. Consider the following function

$$F_0(u) = \frac{1}{p} \|u\|_{\ell_p}^p, \; \nabla F_0(v) = \left\{ |v|^{p-1} \operatorname{sgn}(v) \right\}$$

Let us define the set Ω by

$$\Omega = \{ v \in \mathbb{R}^n \; : \; \nabla F_0(v) \in t\mathcal{B}_{X^*} \},$$

Couple (ℓ^p, X)

$\Omega = \{v \in \mathbb{R}^n \ : \ \nabla F_0(v) \in t\mathcal{B}_{X^*}\}$. There are two cases

Case 1: If $x \in \Omega$

then the optimal decomposition for $L_{p,1}(t, x; \ell_p, X)$ is given by

$$x_{0,\text{opt}} = x \text{ and } x_{1,\text{opt}} = 0$$

Case 2: If $x \notin \Omega$

Theorem

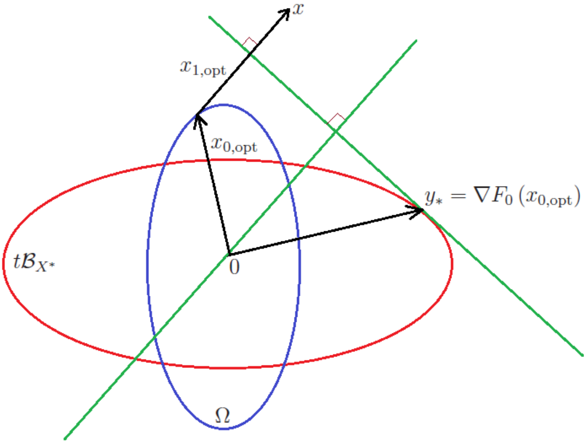
The decomposition $x = x_{0,opt} + x_{1,opt}$ is optimal for $L_{p,1}(t, x; \ell_p, X)$ if and only if

(a) $\|\nabla F_0(x_{0,opt})\|_{X^*} = t$

(b) $\langle x_{1,opt}, \nabla F_0(x_{0,opt})\rangle = t\,\|x_{1,opt}\|_X$.

Geometry of optimal decoposition for couple (ℓ^p, X)

$x_{1,\mathrm{opt}}$ is orthogonal to the supporting hyperplane to $t\mathcal{B}_{X^*}$ at y_*



Case $p = 2$: Couple $\left(\ell^2, X\right)$

$$F_0(u) = \frac{1}{2}\|u\|_{\ell^2}^2, \ \nabla F_0(v) = v$$

The sets Ω and $t\mathcal{B}_{X^*}$ coincide

$$\Omega = t\mathcal{B}_{X^*} = \{u \in \mathbb{R}^n : \ \|u\|_{X^*} \leq t\}$$

Corollary (for $x \notin \Omega$)

$$\|x_{0,opt}\|_{X^*} = t \ and \ \langle x_{0,opt}, x - x_{0,opt}\rangle = t\,\|x - x_{0,opt}\|_X .$$

Case $p = 2$: Couple $\left(\ell^2, X\right)$

Theorem

Let $x_{0,opt}$ be an exact minimizer for $L_{2,1}(t, x; \ell^2, X)$. Then $x_{0,opt}$ is the nearest element of $t\mathcal{B}_{X^*}$ to the point x in the metric of ℓ^2:

$$E\left(t, x; \ell^2, X^*\right) = \inf_{\|x_0\|_{X^*} \leq t} \|x - x_0\|_{\ell^2} = \|x - x_{0,opt}\|_{\ell^2}.$$

Geometry of optimal decoposition for couple (ℓ^2, X)

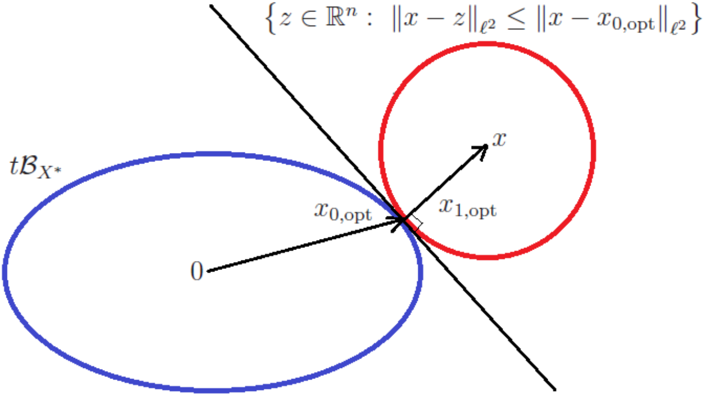


Illustration in the plane

Consider couple $\left(\ell^3, X\right)$ in the plane where the unit ball of X is the rotated ball of ℓ^1 by the rotation matrix

$$R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

for $\theta = 30^\circ$. We have that

$$\|x\|_X = \left\|R_\theta^{-1} x\right\|_{\ell^1} = \left|\frac{\sqrt{3}}{2}x_1 - \frac{1}{2}x_2\right| + \left|\frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2\right|.$$

$$\nabla F_0(u) = \left[|u_1|^2 \operatorname{sgn}(u_1), |u_2|^2 \operatorname{sgn}(u_2)\right].$$

The set Ω can be written as

$$\Omega = \left\{v \in \mathbb{R}^2 : \left\|\left[|v_1|^2 \operatorname{sgn}(v_1), |v_2|^2 \operatorname{sgn}(v_2)\right]^T\right\|_{X^*} \leq t\right\},$$

Illustration in the plane

where the norm in X^* is given by

$$\|y\|_{X^*} = \|R_\theta y\|_{\ell^\infty} = \max\left\{\left|\frac{\sqrt{3}}{2}y_1 + \frac{1}{2}y_2\right|, \left|-\frac{1}{2}y_1 + \frac{\sqrt{3}}{2}y_2\right|\right\}.$$

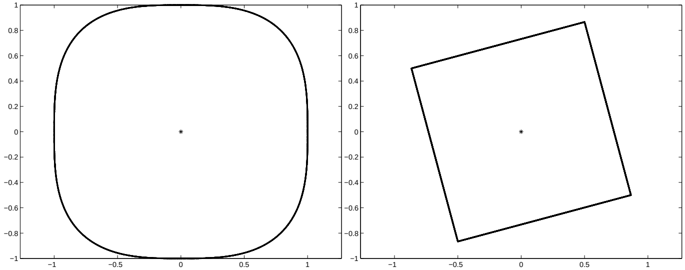
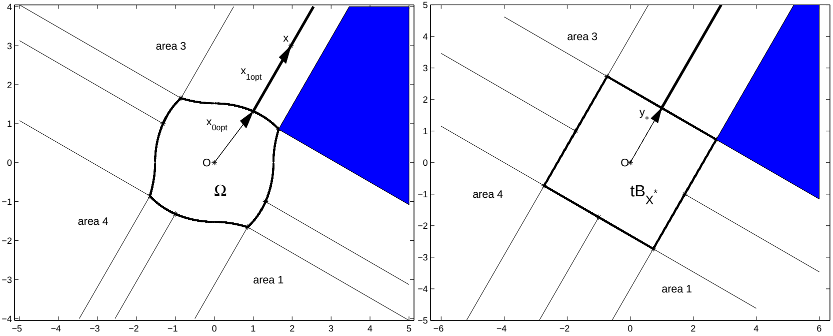


Figure : Unit ball of ℓ^3 (left) and unit ball of X: rotated unit ball of ℓ^1 (right).

Illustration in the plane

Geometry of Optimal Decomposition for the Couple (ℓ_p, X) for $p = 3$, $X = R_\theta(\ell_1)$ and $\theta = 30°$. The set Ω could be of rather complicated structure.



Dual characterization of optimal decomposition

Theorem (**general case**)

Let $x \in X_0 + X_1$, $1 < p_0, p_1 < \infty$ *and let* $t > 0$ *be a fixed parameter. The decomposition* $x = x_{0,opt} + x_{1,opt}$ *is optimal for*

$$L_{p_0,p_1}(t, x; X_0, X_1) = \inf_{x = x_0 + x_1} \left(\frac{1}{p_0} \|x_0\|_{X_0}^{p_0} + \frac{t}{p_1} \|x_1\|_{X_1}^{p_1} \right),$$

if and only if $\exists y_* \in X_0^* \cap X_1^*$ *such that*

$$\begin{cases} \frac{1}{p_0} \|x_{0,opt}\|_{X_0}^{p_0} = \langle y_*, x_{0,opt} \rangle - \frac{1}{p_0'} \|y_*\|_{X_0^*}^{p_0'} ; \\ \frac{t}{p_1} \|x_{1,opt}\|_{X_1}^{p_1} = \langle y_*, x_{1,opt} \rangle - \frac{t}{p_1'} \left\| \frac{y_*}{t} \right\|_{X_1^*}^{p_1'} . \end{cases}$$

Thank you for your attention!