An Adaptive Algorithm for Optimal Control Inverse Problems

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Outline

- Optimal Control as ill-posed and well-posed problems
- Error representation
- Adaptive algorithm
- Numerical tests

Optimal Control

Minimize

$$\int_0^T h\big(X(s),\alpha(s)\big)\,ds + g\big(X(T)\big)$$

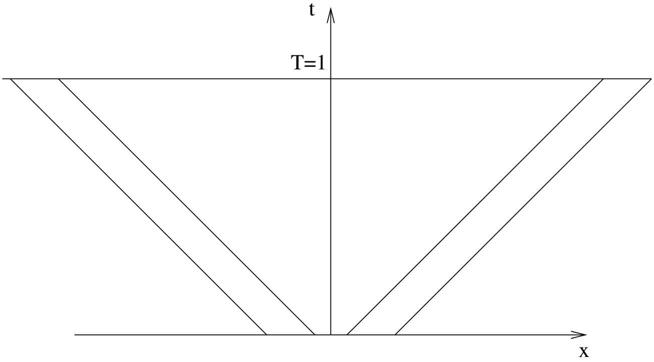
when X solves

$$X'(s) = f\big(X(s),\alpha(s)\big), \quad 0 < s < T$$
$$X(0) = x_0.$$

and $X(s) \in \mathbb{R}^d$, $\alpha(s) \in B$.

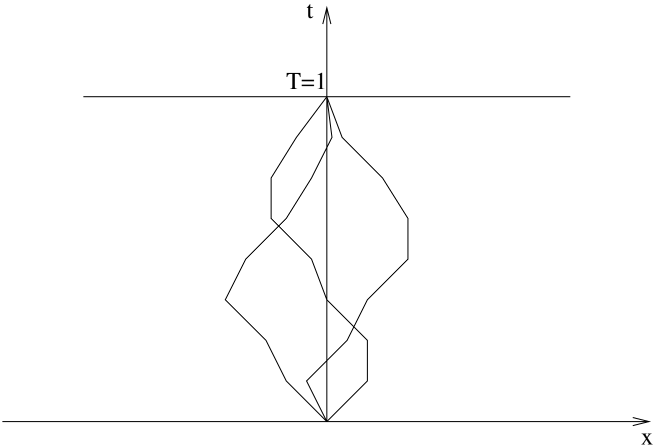
Optimal Control Ill-posed

Ex. Minimize $-|X(1)|$ over solutions $X' = \alpha \in [-1, 1]$.



Optimal Control Ill-posed

Ex. Minimize $|X(1)|$ over solutions $X' = \alpha \in [-1, 1]$.



Optimal Control Well-posed

$$u(x,t) = \inf_{\substack{\alpha \\ X_t = x}} \Big(\int_t^T h(X(s),\alpha(s))ds + g(X(T)) \Big)$$

Hamilton-Jacobi equation:

$$\begin{aligned} u_t + H(u_x, x) &= 0, \\ u(x,T) &= g(x), \end{aligned} \tag{1}$$

where

$$H(\lambda, x) = \min_a \big(\lambda \cdot f(x,a) + h(x,a) \big)$$

(See e.g. L. Evans "Partial Differential Equations".)

Optimal controls are characteristics

Theorem

Assume $H \in C^{1,1}_{loc}(\mathbb{R}^d \times \mathbb{R}^d)$, *$f$, g, h smooth etc., and* (α, X) *optimal control and state variables for the starting position* $(x, t) \in \mathbb{R}^d \times [0, T]$. *Then* $\exists$ *dual path* $\lambda : [t, T] \to \mathbb{R}^d$:

$$\begin{aligned} X'(s) &= H_\lambda\big(\lambda(s), X(s)\big), \\ -\lambda'(s) &= H_x\big(\lambda(s), X(s)\big), \\ \lambda(T) &= g'\big(X(T)\big). \end{aligned}$$

(See e.g. P. Cannarsa, C. Sinestrari, “Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control.”)
This is the Pontryagin principle for the case when the Hamiltonian is differentiable.

Hamilton-Jacobi vs. Pontryagin

	Hamilton-Jacobi	Pontryagin
Global min	+	-
High dimension	-	+

Idea: Use Pontryagin for numerical methods, and Hamilton-Jacobi for theoretical evaluation.

Variational representation of Hamilton-Jacobi

If $u : \mathbb{R}^d \times [0, T] \to \mathbb{R}$ viscosity solution of Hamilton-Jacobi equation (1):

$$u(x,t) = \inf_{\beta} \Big\{ \int_t^T L(\beta, X)dt + g\big(X(T)\big) \big| X'(t) = \beta(t), X(t) = x \Big\},$$

where

$$L(x,\beta) = \sup_{\lambda} \big\{ -\beta \cdot \lambda + H(\lambda, x) \big\}, \text{ and } H(\lambda, x) = \inf_{\beta} \big\{ \lambda \cdot \beta + L(x,\beta) \big\}.$$

(Legendre-type transform)

Representation as discrete minimization

Symplectic Euler

$$
\begin{aligned}
X_{n+1} &= X_n + \Delta t_n H_\lambda(X_n, \lambda_{n+1}),\\
\lambda_n &= \lambda_{n+1} + \Delta t_n H_x(X_n, \lambda_{n+1}),
\end{aligned}
$$

corresponds to minimization of

$$
\sum_{n=0}^{N-1} \Delta t_n L(X_n, \beta_n) + g(X_N) =: \bar{u}(x_0, 0),
$$

where $X_{n+1} = X_n + \Delta t \beta_n$.

Error in approximate value function

Introduce piecewise linear $\bar{X}(t)$ as

$$\bar{X}(t) = X_n + (t - t_n)\beta_n = X_n + (t - t_n)H_\lambda(X_n, \lambda_{n+1}), \qquad t \in (t_n, t_{n+1})$$

$$\begin{aligned}
(\bar{u} - u)(x_0, 0) &= \sum_{n=0}^{N-1} \Delta t_n L(X_n, \beta_n) + \underbrace{g(X_N)}_{u(X_N,T)} - u(x_0, 0) \\
&= \sum_{n=0}^{N-1} \Delta t_n L(X_n, \beta_n) + u(X_N, T) - u(x_0, 0) \\
&= \sum_{n=0}^{N-1} \Delta t_n L(X_n, \beta_n) + \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} u(\bar{X}(t), t)\mathrm{d}t \\
&= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} L(X_n, \beta_n) + u_t(\bar{X}(t), t) + u_x(\bar{X}(t), t) \cdot \beta_n \mathrm{d}t.
\end{aligned}$$

Error in approximate value function

Using

$$\begin{aligned} u_t &= -H(x, u_x), \\ L(X_n, \beta_n) + \lambda_{n+1} \cdot \beta_n &= H(x_n, \lambda_{n+1}), \\ \beta_n &= H_\lambda(x_n, \lambda_{n+1}), \end{aligned}$$

we have

$$\begin{aligned} (\bar{u} - u)(x_0, 0) &= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} L(X_n, \beta_n) + u_t(\bar{X}(t), t) + u_x(\bar{X}(t), t) \cdot \beta_n \mathrm{d}t \\ &= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} H(X_n, \lambda_{n+1}) - H(\bar{X}(t), u_x(\bar{X}(t), t)) \, \mathrm{d}t + \\ &\quad \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \big(u_x(\bar{X}(t), t) - \lambda_{n+1}\big) \cdot H_\lambda(X_n, \lambda_{n+1}) \, \mathrm{d}t \end{aligned}$$

Error in approximate value function

The trapezoidal rule and an assumption on closeness between λ_n and $u_x(x_n, t_n)$ gives

Theorem

Assume $H \in C^2(\mathbb{R}^d \times \mathbb{R}^d)$ and $|\lambda_n - u_x(X_n, \lambda_n)| \leq C\Delta t_{max}$ and the value function u is bounded in $C^3((0,T) \times \mathbb{R}^d)$ (either globally or locally with an extra assumption on X_n convergence). Then

$$\bar{u}(x_0, 0) - u(x_0, 0) = \sum_{n=0}^{N-1} \Delta t_n^2 \rho_n + R, \tag{2}$$

with density

$$\rho_n := -\frac{H_\lambda(X_n, \lambda_{n+1}) \cdot H_x(X_n, \lambda_{n+1})}{2} \tag{3}$$

and the remainder term $|R| \leq C\Delta t_{max}^2$, for some constant C.

Algorithm (Adaptivity (Numer. Math. 03 M. S. T. Z.))

Choose the error tolerance TOL, the initial grid $\{t_n\}_{n=0}^N$, *the parameters s and M, and repeat the following points:*

1. *Calculate* $\{(X_n, \lambda_n)\}_{n=0}^N$.
2. *Calculate error densities* $\{\rho_n\}_{n=0}^N$ *and the corresponding approximate error densities*
$$\bar{\rho}_n := \operatorname{sgn}(\rho_n)\max(|\rho_n|, \sqrt{\Delta t_{max}}).$$
3. *Break if*
$$\max_n \bar{r}_n < \frac{TOL}{N}.$$
where the error indicators are defined by $\bar{r}_n := |\bar{\rho}_n| \Delta t_n^2$.
4. *Traverse through the mesh and subdivide an interval* (t_n, t_{n+1}) *into M parts if*
$$\bar{r}_n > s\frac{TOL}{N}.$$
5. *Update N and* $\{t_n\}_{n=0}^N$ *to reflect the new mesh.*

Introduce a constant $c = c(t)$, such that

$$c \leq |\frac{\bar{\rho}(t)[\text{parent}(n,k)]}{\bar{\rho}(t)[k]}| \leq c^{-1}, \quad c \leq |\frac{\bar{\rho}(t)[k-1]}{\bar{\rho}(t)[k]}| \leq c^{-1}, \tag{4}$$

holds for all time steps $t \in \Delta t_n[k]$ and all refinement levels k.

Theorem

[Stopping] Assume that c satisfies (4) for the time steps corresponding to the maximal error indicator on each refinement level, and that

$$M^2 > c^{-1}, \; s \leq \frac{c}{M}. \tag{5}$$

Then each refinement level either decreases the maximal error indicator with the factor

$$\max_n \bar{r}_n[k+1] \leq \frac{c^{-1}}{M^2} \max_n \bar{r}_n[k],$$

or stops the algorithm.

Theorem

[Accuracy] The adaptive algorithm satisfies

$$\limsup_{TOL\to 0^+} \left(TOL^{-1} |u(x_0,0) - \bar{u}(x_0,0)| \right) \leq 1.$$

Theorem

[Efficiency] Assume that $c = c(t)$ satisfies (4) for all time steps at the final refinement level, and that all initial time steps have been divided when the algorithm stops. Then there exists a constant $C > 0$, bounded by $M^2 s^{-1}$, such that the final number of adaptive steps N, of the algorithm 0.3, satisfies

$$TOL\ N \leq C \|\frac{\bar{\rho}}{c}\|_{L^{\frac{1}{2}}} \leq \|\bar{\rho}\|_{L^{\frac{1}{2}}} \max_{0\leq t\leq T} c(t)^{-1},$$

and $\|\bar{\rho}\|_{L^{\frac{1}{2}}} \to \|\tilde{\rho}\|_{L^{\frac{1}{2}}}$ asymptotically as $TOL \to 0^+$.

Numerical Tests

Ex. (From PROPT Manual, by Rutquist, Edvall). Minimize

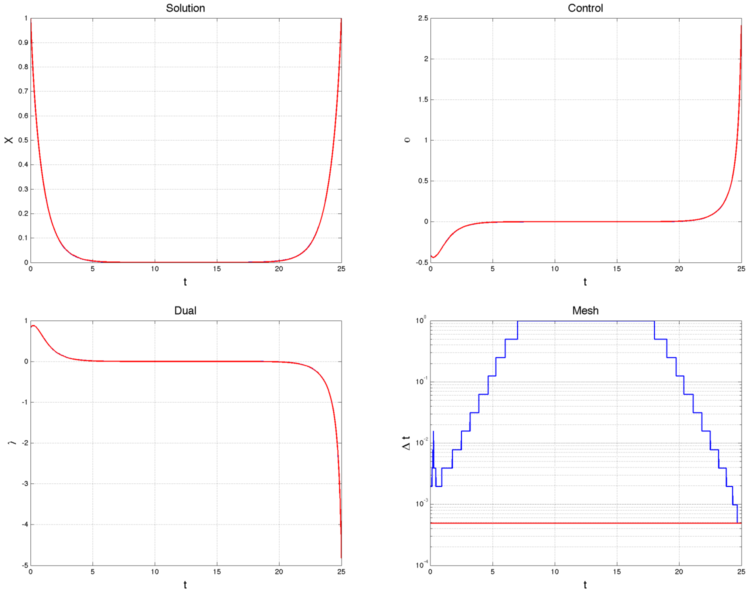
$$\int_0^{25} X(t)^2 + \alpha(t)^2 \mathrm{d}t + \gamma(X(25) - 1)^2,$$

subject to

$$X'(t) = -X(t)^3 + \alpha(t), \qquad 0 < t \leq 25,$$
$$X(0) = 1.$$

for some large $\gamma > 0$. The Hamiltonian is then given by

$$H(x, \lambda) := \min_{\alpha} \left\{ -\lambda x^3 + \lambda\alpha + x^2 + \alpha^2 \right\} = -\lambda x^3 - \lambda^2/4 + x^2.$$


Solution
X
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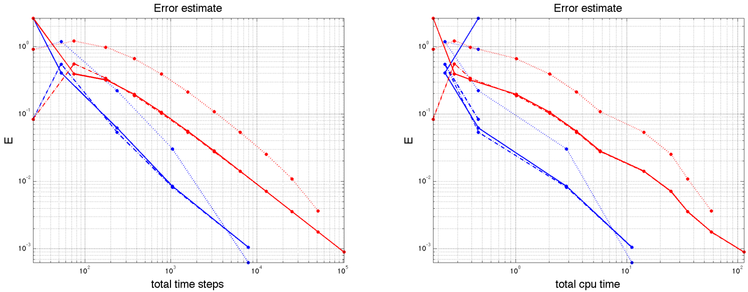


Figure: The solid lines (red = uniform, blue = adaptive) indicate the error estimate, and the dotted lines the difference between the value function and the value function using the finest mesh for the uniform refinement.

Common problem: Non-smooth Hamiltonian

Ex. Minimize

$$\int_0^1 X(t)^{10} \mathrm{d}t,$$

subject to

$$X'(t) = \alpha(t) \in [-1, 1], \quad 0 < t \leq T,$$
$$X(0) = 0.5.$$

The Hamiltonian is then non-smooth

$$H(x, \lambda) := \min_{\alpha \in [-1,1]} \left\{ \lambda\alpha + x^{10} \right\} = -|\lambda| + x^{10},$$

but can be regularized by

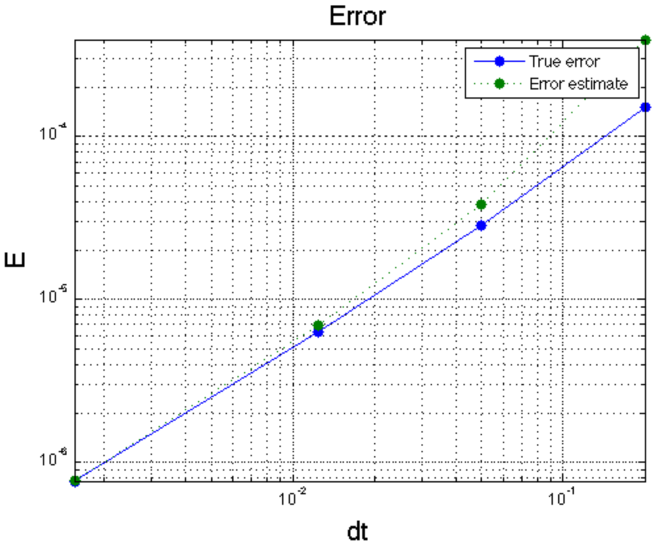
$$H_\delta(x, \lambda) := -\sqrt{\lambda^2 + \delta^2} + x^{10},$$

for some small $\delta > 0$.

Changing the Hamiltonian H to H^δ, with $|H - H^\delta| = \mathcal{O}(\delta)$ introduces an error of order δ in the value function u. However, the remainder term R in the error representation contains second derivatives of H, and $\partial_{\lambda\lambda} H^\delta = \mathcal{O}(\delta^{-1})$.

On the other hand we have an a priori error estimate

$$|u - \bar{u}^\delta| = \mathcal{O}(\delta + \Delta t).$$


Error
True error
Error estimate
E
10^{-4}
10^{-5}
10^{-6}
10^{-2}
10^{-1}
dt

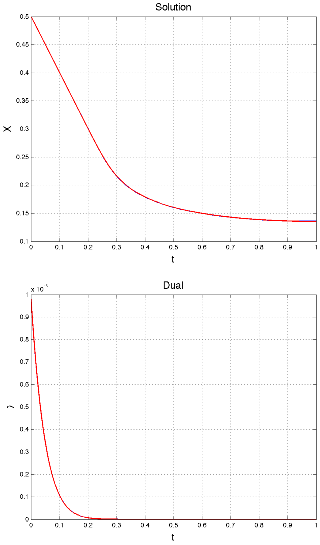


Figure: $\delta = 10^{-6}$ and $TOL = 10^{-6}$. The blue and red lines indicate solutions from adaptive and uniform meshes, respectively, corresponding to the lower right plot.

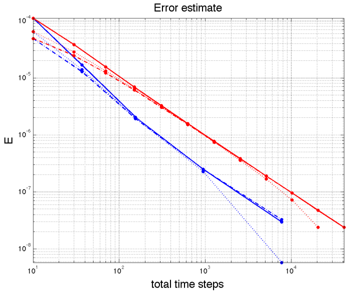


Figure: The solid lines indicate the error estimate while the dotted lines indicate the difference between the value function and the value function using the finest mesh for the uniform refinement.

Ex. (Fuller's problem)

(From PROPT Manual, by Rutquist, Edvall). Minimize

$$\int_0^1 X_1(t)^2 \mathrm{d}t + \gamma(X_1(1) - 0.01)^2 + \gamma X_2(1)^2,$$

subject to

$$X_1'(t) = X_2(t), \qquad 0 < t \le T, \qquad X_1(0) = 0.01,$$
$$X_2'(t) = -\alpha(t), \qquad X_2(0) = 0,$$

and

$$|\alpha(t)| \le 1,$$

for some large $\gamma > 0$. The Hamiltonian is then non-smooth

$$H(x_1, x_2, \lambda_1, \lambda_2) := \min_{\alpha \in [-1,1]} \left\{ \lambda_1 x_2 - \lambda_2 \alpha + x_1^2 \right\} = \lambda_1 x_2 - |\lambda_2| + x_1^2,$$

but can be regularized by

$$H_\delta(x_1, x_2, \lambda_1, \lambda_2) := \lambda_1 x_2 - \sqrt{\lambda_2^2 + \delta^2} + x_1^2,$$

for some small $\delta > 0$.

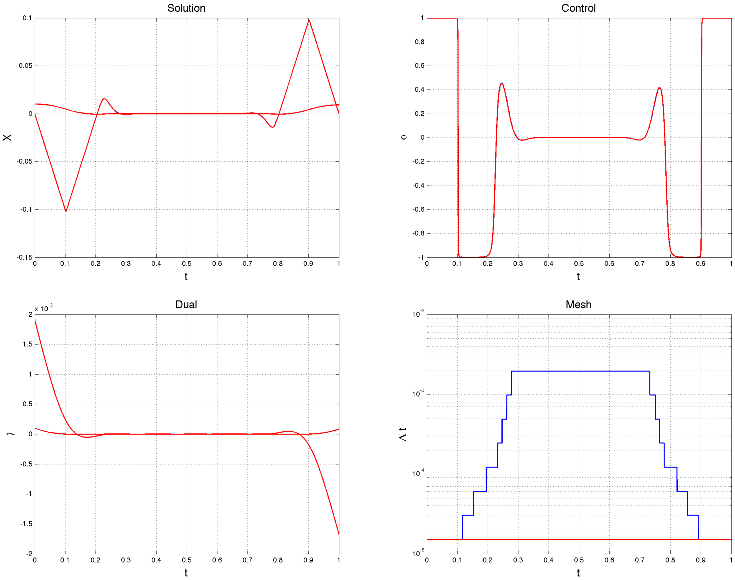


Figure: $\gamma = 1$, $\delta = 10^{-7}$ and $TOL = 10^{-6}$. The blue and red lines here indicate solutions from adaptive and uniform meshes, respectively.

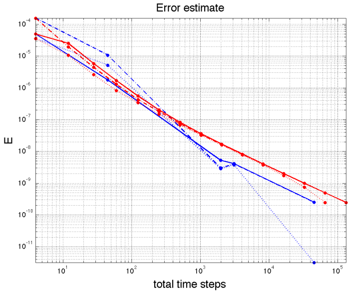


Figure: The solid lines indicate the error estimate and the dotted lines indicate the difference between the value function and the value function using the finest mesh for the uniform refinement.

Allen-Cahn Ex.

$$u(\varphi_0, t_0) = \min_{\alpha} \int_{t_0}^{T} h\big(\alpha(t)\big)\,dt + g\big(\varphi(T)\big),$$

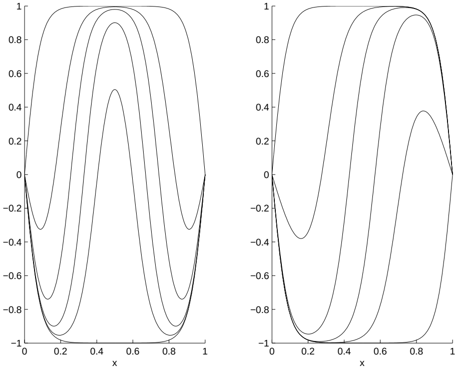
$$h(\alpha) = ||\alpha||^2_{L^2(0,1)}/2, \quad g(\varphi_T) = K||\varphi_T - \varphi_-||^2_{L^2(0,1)}$$

and φ solves

$$\varphi_t = \varepsilon\varphi_{xx} - \varepsilon^{-1}V'(\varphi) + \alpha, \quad \varphi(t_0) = \varphi_0.$$

Then

$$u(\varphi_0, 0) - \bar{u}(\varphi_0, 0) = \mathcal{O}(\Delta t + (\Delta x)^2).$$



1
0.8
0.6
0.4
0.2
0
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−0.8
−1
0
0.2
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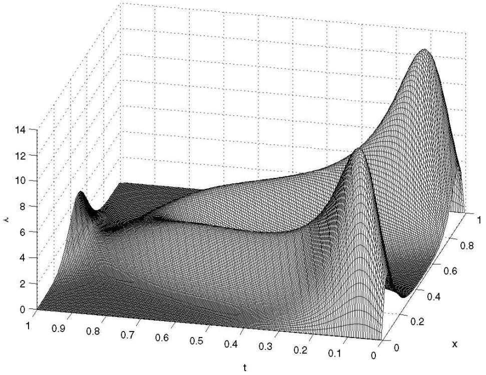
0.6
0.8
1
x


14
12
10
8
6
4
2
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y
1
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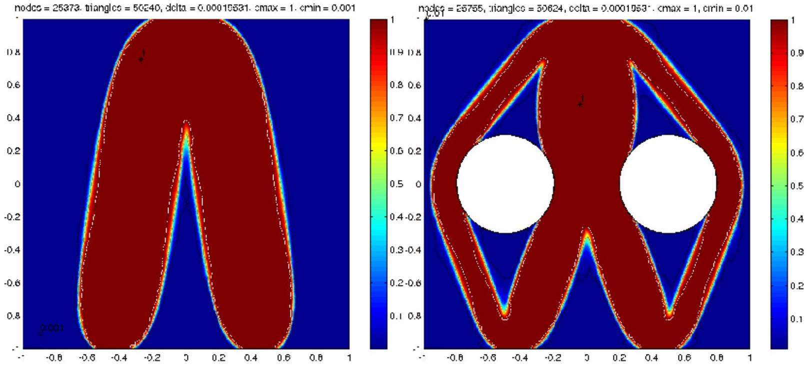
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
t
1
0.8
0.6
0.4
0.2
0
x

Ex: Computation of optimal designs. Find $\sigma : \Omega \to \{\sigma_-, \sigma_+\}$

$$\operatorname{div}(\sigma \partial_x \varphi(x)) = 0 \quad x \in \Omega, \qquad \sigma \frac{\partial \varphi}{\partial n}\Big|_{\partial\Omega} = I$$

$$\min_\sigma \Big(\int_{\partial\Omega} I \varphi ds + \eta \int_\Omega \sigma dx \Big).$$



Regularized Hamiltonian H^δ, with $\delta \to 0$ possible.

Conclusions

- Error representation for value function associated with optimal control problems using discretization of Hamiltonian system.
- May be used for adaptive algorithms.
- Adaptivity natural since some iterative method must be used to solve the initial-terminal time boundary value problem. A solution on a grid level may be used as initial guess for the solution on the next level.
- Seems adaptivity is not essential for problems with discontinuous controls.
- Seems that the computable leading order term in the error representation is dominant even in cases where the Hamiltonian has large second order derivatives. Open problem to show this theoretically.