NUMERICAL CONVERGENCE OF BREGMAN ITERATES FOR THE HYBRID INVERSE CONDUCTIVITY PROBLEM

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1 PROBLEM FORMULATION

$$\nabla \cdot (\sigma \nabla v) = 0, \quad v|_{\partial\Omega} = f$$

$$\nabla \cdot \left(\frac{|J|}{|\nabla v|} \nabla v \right) = 0, \quad v|_{\partial\Omega} = f$$

$$\text{argmin}\{ \textstyle\int_\Omega |J||\nabla v| : v \in H^1(\Omega) \ \ v|_{\partial\Omega} = f \}.$$

GIVEN $|J|$, FIND A MINIMIZER OF THIS PROBLEM

OPEN QUESTION: HOW TO CONSTRUCT CONVERGENT MINIMIZING SEQUENCES?

MAIN DIFFICULTY: THE MINIMIZING FUNCTIONAL IS NOT DIFFERENTIABLE

KEY IDEA: CONSTRUCT BREGMAN ITERATES

2 REFERENCES

L. BREGMAN, The relaxation method of finding the common points of convex sets and its applications to the solution of problems of convex programming. *USSR Comput.Math. anf Math.Phys.*, **7** (1967), 200-17.

J. ECKSTEIN and D.P. BERTSEKAS, On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators. *Math. Programming*, **55** (1992), 293-318.

S. SETZER, Split Bregman Algorithm, Douglas-Rachford Splitting and Frame Shrinkage, Proc. of the Second International Conference on Scale Space Methods and Variational Methods in Computer Visio, Springer, 2009.

A. MORADIFAM, A. NACHMAN, and A. TIMONOV, A convergent algorithm for the hybrid problem of reconstructing conductivity from minimal interior data. *Inverse Problems*, **28** (2012).

5 ALTERNATING SPLIT BREGMAN ALGORITHM

INITIALIZE:

$u_f \in H^1(\Omega)$ with $u_f|_{\partial\Omega} = f$, $b^0, d^0 \in (L^2(\Omega))^n$

STEP 1: FOR $k \geq 1$ SOLVE THE PROBLEM

$$\Delta u^{k+1} = \nabla \cdot (d^k(x) - b^k(x)), \quad u^{k+1}|_{\partial\Omega} = 0.$$

STEP 2: UPDATE

$$d^{k+1}(x) = \begin{cases} \max\{|z^{k+1}(x)| - \frac{|J|}{\alpha}, 0\} \frac{z^{k+1}(x)}{|z^{k+1}(x)|} - \nabla u_f(x) & \text{if } |z^{k+1}(x)| \neq 0, \\ -\nabla u_f(x) & \text{if } |z^{k+1}(x)| = 0. \end{cases}$$

WHERE

$z^{k+1}(x) = \nabla u^{k+1}(x) + \nabla u_f(x) + b^k(x).$

STEP 3: UPDATE

$$b^{k+1}(x) = b^k(x) + \nabla u^{k+1}(x) - d^{k+1}(x).$$

6 CONVERGENCE RESULT

THEOREM. *Let Ω be a bounded region in R^n with a connected $C^{1,\alpha}$ boundary. Assume that $|J| > 0$ a.e. in Ω, where $J \in (L^2(\Omega))^n$ is the current density vector field determined by an unknown conductivity $\sigma \in C^\alpha(\Omega)$ and voltage $f \in H^{1/2}$ on $\partial\Omega$. Then*

$$b^k \rightharpoonup -J/\alpha,$$

$$d^k + \nabla u_f \rightharpoonup \nabla \bar{u}, \text{ and}$$

$$u^k + u_f \rightharpoonup \bar{u},$$

where the voltage potential $\bar{u}$ is the unique minimizer of the least gradient problem.

Finite-dimensional case: Eckstein-Bertsekas (1992), reproved by Setzer (2009)

Infinite-dimensional case: Moradifam-Nachman (2012)

Evans-Spruck Regularization

In a simple iteration algorithm

$$\sigma^{k+1} = \frac{|J|}{\sqrt{|\nabla v^k|^2 + \varepsilon_k^2}}$$

Conjecture For any positive monotone-decreasing sequence ε_k
(e.g.,

$$\varepsilon_k = \gamma \cdot \varepsilon_{k-1}, \quad \gamma \in (0,1))$$

the convergence takes place

$$\sigma^k \to \bar{\sigma} \quad \text{as} \quad k \to \infty$$

QRM (TIkhonov's regularization)

By analogy with Ladyzhenskaya and Lions, consider the regularized least gradient functional

$$\operatorname{argmin}\{\int_\Omega \left(|J||\nabla v| + \frac{\alpha}{2}|\nabla v|^2\right) : v \in H^1(\Omega).\ \ v|_{\partial\Omega} = f\}$$

The Euler-Lagrange equation

$$\alpha \cdot \nabla^2 v + \nabla \cdot \left(\frac{|J|}{|\nabla v|}\nabla v\right) = 0, \quad v|_{\partial\Omega} = f.$$

Conjecture Let v_α be a solution to the Euler-Lagrange equation for the fixed $\alpha > 0$. Then the sequence $\{v_\alpha\}$ is minimizing for the least gradient functional.

8 NUMERICAL STUDY

MODEL PROBLEM

SIMULATE THE CONDUCTIVITY DISTRIBUTION

Figure 1: The original conductivity distribution used in the data simulation.

SIMULATE THE INTERNAL DATA $|J|$

$$|J| = \sigma(x)|\nabla v|,$$

$$v = u_h + u,$$

$$-\nabla \cdot \sigma \nabla u = \nabla \cdot \sigma \nabla u_h, \quad u_{|\partial\Omega} = 0,$$

$$\nabla^2 u_h = 0, \quad u_{h|\partial\Omega} = f,$$

RECONSTRUCTIONS BY ALTERNATING SPLIT BREGMAN ALGORITHM: THE ALMOST TWO-TO-ONE BC

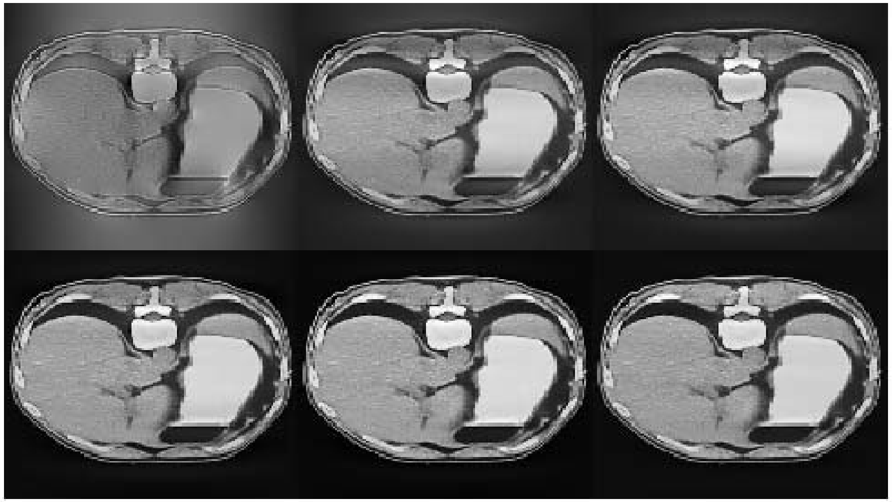

Figure 2: Reconstruction using the altrnating split Bregman algorithm with $N = 1, 5, 10, 30, 50, 100$ iterations (show from the left upper corner to the right lower corner) for the almost two-to-one boundary condition $f = y$.

Figure 3: Conductivity reconstructed using the alternating split Bregman algorithm with the optimal number of iterates 60 for the almost two-to-one boundary condition $f(x, y) = y$.

RECONSTRUCTIONS BY THE SIMPLE ITERATION ALGORITHM: THE ALMOST TWO-TO-ONE BC

$$u_0 = u_h, \sigma_1 = \frac{|J|}{\nabla u_0},$$

$$-\Delta u = \nabla \cdot (\sigma_k \nabla u_h), \quad u_{|\partial\Omega} = 0,$$

$$v_k = u + u_h,$$

$$\sigma_{k+1} = \frac{|J|}{|\nabla v_k|}.$$

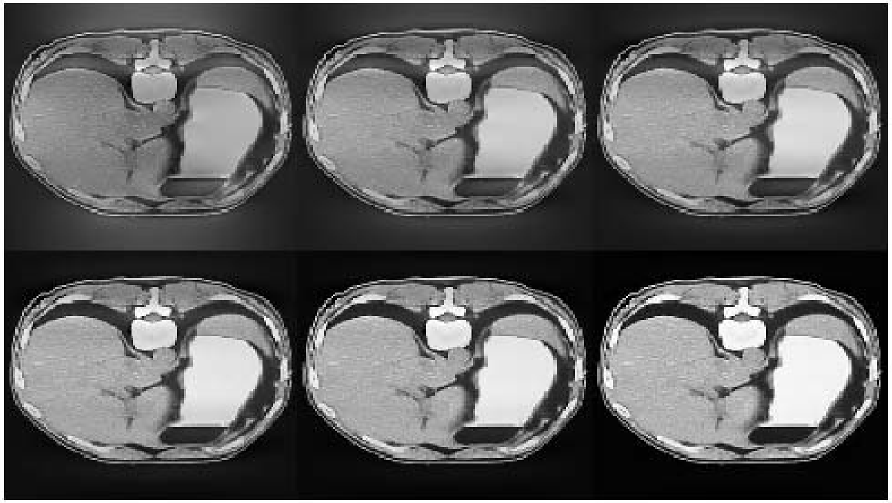

Figure 4: Reconstruction using the simple iteration algorithm with $N = 1, 5, 10, 30, 50, 100$ iterations (show from the left upper corner to the right lower corner) for the almost two-to-one boundary condition $f = y$.

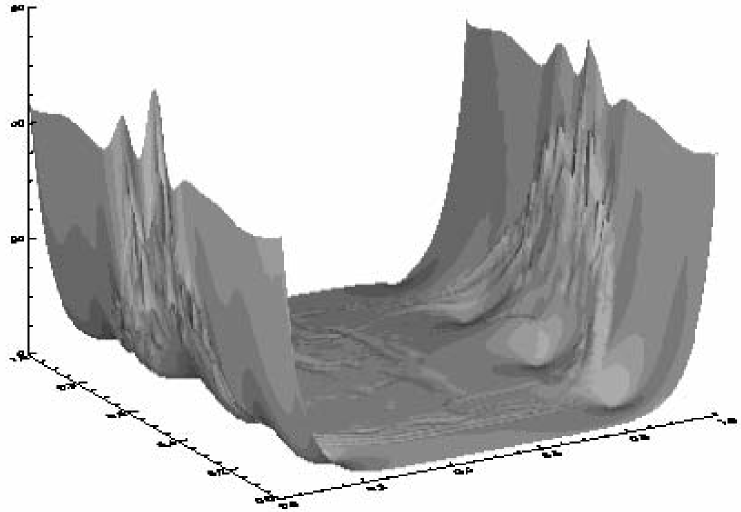

Figure 5: Magnitude of the current density $|J|$ for the non two-to-one boundary data $f(x, y) = y + 2\sin(7\pi y)$.

RECONSTRUCTIONS BY ALTERNATING SPLIT BREGMAN ALGORITHM: THE NON-TWO-TO-ONE BC

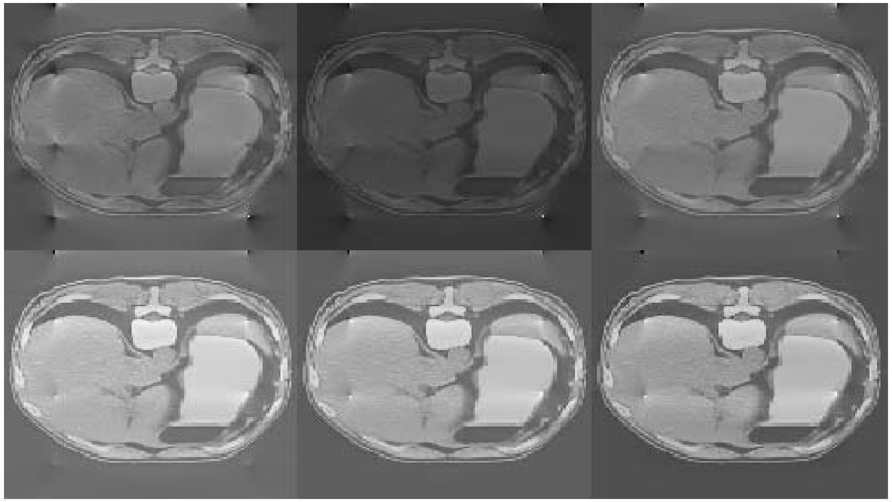

Figure 6: Reconstruction using the altrnating split Bregman algorithm with $N = 1, 5, 10, 30, 50, 100$ iterations (show from the left upper corner to the right lower corner) for the non- two-to-one boundary condition $f(x, y) = y + 2\sin(7\pi y)$.

RECONSTRUCTIONS BY REGULARISED SIMPLE ITERATION ALGORITHM: THE NON-TWO-TO-ONE BC

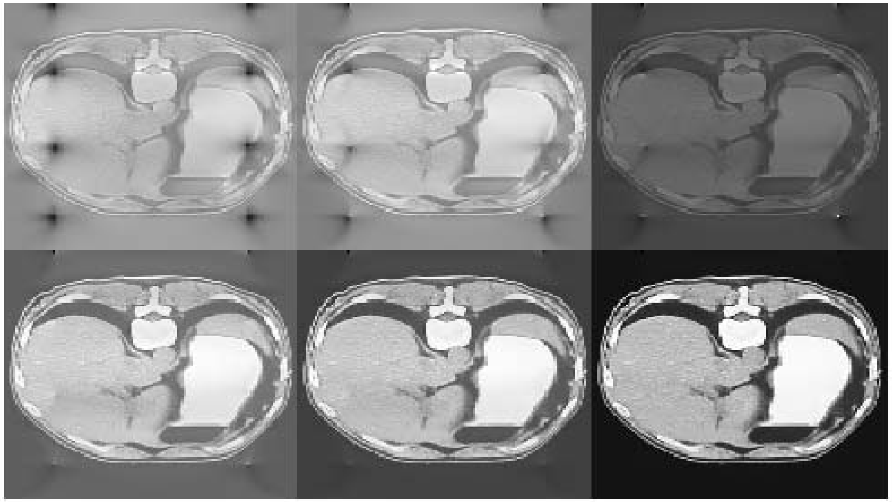

Figure 7: Reconstruction using the regularised simple iteration algorithm with $N = 1, 5, 10, 30, 50, 100$ iterations (show from the left upper corner to the right lower corner) for the non- two-to-one boundary condition $f(x, y) = y + 2\sin(7\pi y)$.

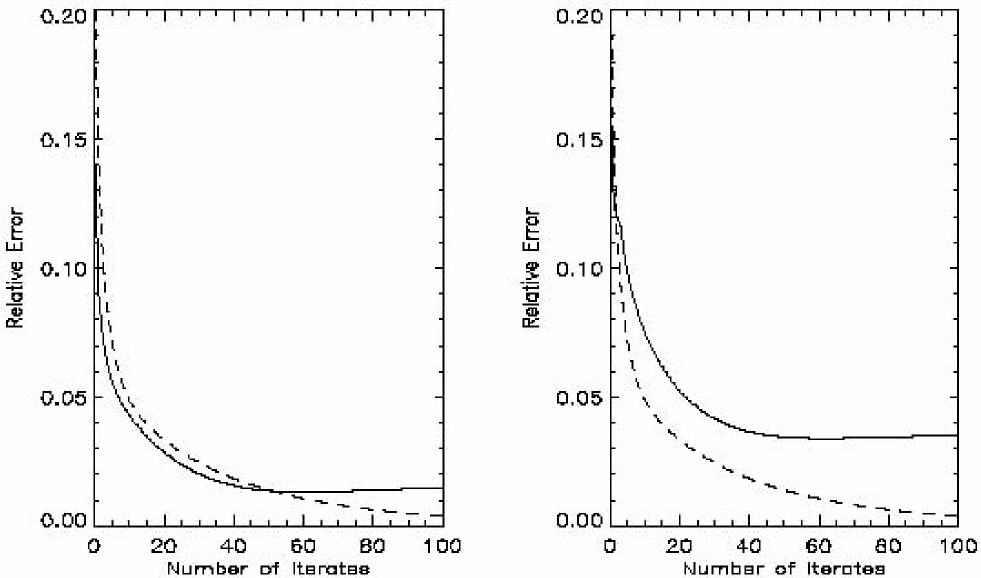

Figure 8: Rate of convergence for alternating split Bregman (Solid) and simple iterations (Dashed) for two-to-one boundary data $f(x, y) = y$ (left) and non two-to-one data $f(x, y) = y + 2\sin(7\pi y)$.

ROBUSTNESS

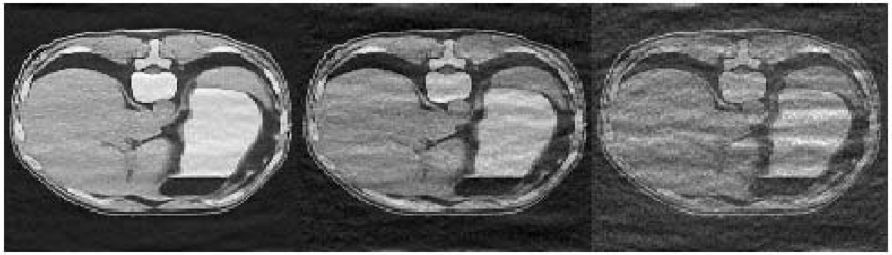

Figure 9: Low noise (left), moderate noise (middle), and higher noise.