Inference in k-Exchangeable Multivariate Models

Miguel Fonseca   Anuradha Roy

LinStat'2014

Model

Formulation

$$\underset{p\times r}{\boldsymbol{Y}_i} = \underset{p\times m}{\boldsymbol{B}}\,\underset{m\times r}{\boldsymbol{X}'} + \underset{p\times r}{\boldsymbol{E}_i},$$

where $\boldsymbol{X}$ is the design matrix

$$\begin{aligned}
\mathrm{Vec}(\underset{p\times r}{\boldsymbol{Y}_i}) &= \mathrm{Vec}(\underset{p\times m}{\boldsymbol{B}}\,\underset{m\times r}{\boldsymbol{X}'}) + \mathrm{Vec}(\underset{p\times r}{\boldsymbol{E}_i}),\\
\underset{pr\times 1}{\boldsymbol{y}_i} &= (\underset{r\times m}{\boldsymbol{X}} \otimes \boldsymbol{I}_p)\mathrm{Vec}(\underset{p\times m}{\boldsymbol{B}}) + \underset{pr\times 1}{\boldsymbol{e}_i},\\
\underset{pr\times 1}{\boldsymbol{y}_i} &= (\underset{r\times m}{\boldsymbol{X}} \otimes \boldsymbol{I}_p)\underset{pm\times 1}{\boldsymbol{\beta}} + \underset{pr\times 1}{\boldsymbol{e}_i},
\end{aligned}$$

where $\underset{pr\times 1}{\boldsymbol{e}_i} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{V})$ for $i = 1, \ldots, n$

Model

Parameters

Expectation

$$
\begin{aligned}
\mathbb{E}[\boldsymbol{Y}_i] &= \boldsymbol{B}\boldsymbol{X}', \\
\mathbb{E}[\mathrm{Vec}\,\boldsymbol{Y}_i)] &= \mathrm{Vec}(\boldsymbol{B}\boldsymbol{X}'), \\
\mathbb{E}[\boldsymbol{y}_i] &= (\boldsymbol{X} \otimes \boldsymbol{I}_p)\mathrm{Vec}(\boldsymbol{B}), \\
\mathbb{E}[\boldsymbol{y}_i] &= (\boldsymbol{X} \otimes \boldsymbol{I}_p)\beta,
\end{aligned}
$$

Model

Parameters

Covariance Matrix

$$\mathbb{V}[\boldsymbol{y}_i] = \boldsymbol{V} = \sum_{j=0}^{w} \boldsymbol{Q}_j \otimes \boldsymbol{\Gamma}_j,$$

with $\boldsymbol{\Gamma}_j$ positive definite matrices.

Orthogonality

Covariance Structure Conditions

- $\boldsymbol{Q}_j$, $j = 0, \ldots, j = w$, are orthogonal projection matrices
- $\boldsymbol{Q}_{j_1}\boldsymbol{Q}_{j_2} = \boldsymbol{0}$, for $j_1 \neq j_2$

Mean Vector Structure Conditions

- $\boldsymbol{P_X}\boldsymbol{Q}_j = \boldsymbol{Q}_j\boldsymbol{P_X}$, where $\boldsymbol{P_X} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$.

Model Properties

- Multivariate Commutative Orthogonal Block Structure – COBS
- $\boldsymbol{V}^{-1} = \sum_{j=1}^{w} \boldsymbol{Q}_j \otimes \boldsymbol{\Gamma}_j^{-1}$
- $|\boldsymbol{V}| = \prod_{j=1}^{w} |\boldsymbol{\Gamma}_j|^{g_j}$, where $g_j = \mathrm{rank}(\boldsymbol{Q}_j)$

Lemma

$(\boldsymbol{I} - \boldsymbol{P}_{\boldsymbol{X}})$ and $\boldsymbol{Q}_j$ are commutative, i.e.,

$$(\boldsymbol{I} - \boldsymbol{P}_{\boldsymbol{X}})\boldsymbol{Q}_j = \boldsymbol{Q}_j(\boldsymbol{I} - \boldsymbol{P}_{\boldsymbol{X}}),$$

Doubly Exchangeable Structure

$$\boldsymbol{y} = \begin{pmatrix} \boldsymbol{y}_1 \\ \vdots \\ \boldsymbol{y}_v \end{pmatrix}, \quad \text{where} \quad \boldsymbol{y}_t = \begin{pmatrix} \boldsymbol{y}_{t1} \\ \vdots \\ \boldsymbol{y}_{tu} \end{pmatrix}, \quad \text{with} \quad \boldsymbol{y}_{ts} = \begin{pmatrix} \boldsymbol{y}_{ts1} \\ \vdots \\ \boldsymbol{y}_{tsm} \end{pmatrix}$$

for $s = 1, \ldots, u,\ t = 1, \ldots, v.$

$$\boldsymbol{\Gamma}_{\boldsymbol{y}} = \mathbf{I}_{vu} \otimes (\mathbf{U}_0 - \mathbf{U}_1) + \mathbf{I}_v \otimes \mathbf{J}_u \otimes (\mathbf{U}_1 - \mathbf{W}) + \mathbf{J}_{vu} \otimes \mathbf{W},$$

$$\text{Cov}\left[\boldsymbol{y}_{ts}; \boldsymbol{y}_{t^*s^*}\right] = \begin{cases} \mathbf{U}_0 & \text{if} \quad t = t^* \quad \text{and} \quad s = s^*, \\ \mathbf{U}_1 & \text{if} \quad t = t^* \quad \text{and} \quad s \neq s^*, \\ \mathbf{W} & \text{if} \quad t \neq t^*, \end{cases}$$

Definition

Let $\boldsymbol{y}_r$ be an $muv-$variate partitioned real-valued random vector $\boldsymbol{y}_r = (\boldsymbol{y}'_{r,1}, \ldots, \boldsymbol{y}'_{r,v})'$, where $\boldsymbol{y}_{r,t} = (\boldsymbol{y}'_{r,t1}, \ldots, \boldsymbol{y}'_{r,tu})'$ for $t = 1, \ldots, v$, and $\boldsymbol{y}'_{r,ts} = (y_{r,ts,1}, \ldots, y_{r,ts,m})'$ for $s = 1, \ldots, u$. Let $\boldsymbol{\mu}_{\boldsymbol{x}} \in \Re^{puv}$ be the mean vector, and $\boldsymbol{V}$ be the $(puv \times puv)-$dimensional partitioned covariance matrix $\boldsymbol{V} = \text{Cov}[\boldsymbol{y}] = (\boldsymbol{\Gamma}_{\boldsymbol{y}_{r,t}, \boldsymbol{y}_{r,t^*}}) = (\boldsymbol{\Gamma}_{r,tt^*})$, where $\boldsymbol{\Gamma}_{r,tt^*} = \text{Cov}[\boldsymbol{y}_{r,t}, \boldsymbol{y}_{r,t^*}]$ for $t, t^* = 1, \ldots, v$, is given by

$$\boldsymbol{V} = \left[\begin{array}{cccc|cccc|c|cccc}
\boldsymbol{U}_0 & \boldsymbol{U}_1 & \cdots & \boldsymbol{U}_1 & \boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} & \cdots & \boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} \\
\boldsymbol{U}_1 & \boldsymbol{U}_0 & \cdots & \boldsymbol{U}_1 & \boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} & \cdots & \boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\boldsymbol{U}_1 & \boldsymbol{U}_1 & \cdots & \boldsymbol{U}_0 & \boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} & \cdots & \boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} \\
\hline
\boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} & \boldsymbol{U}_0 & \boldsymbol{U}_1 & \cdots & \boldsymbol{U}_1 & \cdots & \boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} \\
\boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} & \boldsymbol{U}_1 & \boldsymbol{U}_0 & \cdots & \boldsymbol{U}_1 & \cdots & \boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} & \boldsymbol{U}_1 & \boldsymbol{U}_1 & \cdots & \boldsymbol{U}_0 & \cdots & \boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} \\
\hline
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\hline
\boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} & \boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} & \cdots & \boldsymbol{U}_0 & \boldsymbol{U}_1 & \cdots & \boldsymbol{U}_1 \\
\boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} & \boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} & \cdots & \boldsymbol{U}_1 & \boldsymbol{U}_0 & \cdots & \boldsymbol{U}_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} & \boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} & \cdots & \boldsymbol{U}_1 & \boldsymbol{U}_1 & \cdots & \boldsymbol{U}_0
\end{array}\right],$$

where $\boldsymbol{U}_0$ is a positive definite symmetric $p \times p$ matrix, and $\boldsymbol{U}_1$ and $\boldsymbol{W}$ are symmetric $p \times p$ matrices. The variance-covariance matrix $\boldsymbol{\Gamma}_{\boldsymbol{y}}$ is then said to have a jointly equicorrelated covariance structure with equicorrelation parameters $\boldsymbol{U}_0, \boldsymbol{U}_1$ and $\boldsymbol{W}$. The matrices $\boldsymbol{U}_0, \boldsymbol{U}_1$ and $\boldsymbol{W}$ are all unstructured.

Model

$$\underset{m\times u(v)}{\boldsymbol{Y}} = \underset{m\times 1}{\boldsymbol{\alpha}}\underset{1\times u(v)}{\boldsymbol{1}'} + \underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times u(v)}{\boldsymbol{T}} + \underset{m\times u(v)}{\boldsymbol{E}},$$

$$\underset{u(v)\times m}{\boldsymbol{Y}'} = \underset{u(v)\times 1}{\boldsymbol{1}}\underset{1\times m}{\boldsymbol{\alpha}'} + \underset{u(v)\times r-1}{\boldsymbol{T}'}\underset{r-1\times m}{\boldsymbol{\gamma}} + \underset{u(v)\times m}{\boldsymbol{E}'},$$

Lemma

Let $\boldsymbol{\Gamma} = \underset{v\times v}{\boldsymbol{C}'} \otimes \boldsymbol{I}_{mu}$ *and* $\boldsymbol{\Gamma}^{\bullet} = \boldsymbol{I}_v \otimes (\underset{u\times u}{\boldsymbol{C}^{*\prime}} \otimes \boldsymbol{I}_m)$ *where* $\boldsymbol{C}$ *and* $\boldsymbol{C}^*$ *are orthogonal matrices whose first columns are proportional to* $\mathbf{1}$*'s. Let* $\boldsymbol{\Gamma}_{\boldsymbol{y}}$ *be a jointly equicorrelated covariance matrix as in equation* (1) *of Def. 1, then* $\boldsymbol{\Gamma}^{\bullet}\boldsymbol{\Gamma}(\boldsymbol{\Gamma}_{\boldsymbol{y}})\boldsymbol{\Gamma}'\boldsymbol{\Gamma}^{\bullet\prime}$ *is a diagonal matrix as follows:*

$$\boldsymbol{\Gamma}^{\bullet}\boldsymbol{\Gamma}(\boldsymbol{\Gamma}_{\boldsymbol{y}})[\boldsymbol{\Gamma}'\boldsymbol{\Gamma}^{\bullet\prime} = \begin{bmatrix} \boldsymbol{\Delta}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_{u-1}\otimes\boldsymbol{\Delta}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Delta}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{I}_{u-1}\otimes\boldsymbol{\Delta}_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\Delta}_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{I}_{u-1}\otimes\boldsymbol{\Delta}_1 \end{bmatrix},$$

where

$$\begin{aligned} \boldsymbol{\Delta}_1 &= \mathbf{U}_0 - \mathbf{U}_1, \\ \boldsymbol{\Delta}_2 &= \mathbf{U}_0 + (u-1)\,\mathbf{U}_1 - u\mathbf{W} = (\mathbf{U}_0 - \mathbf{U}_1) + u\,(\mathbf{U}_1 - \mathbf{W}), \quad \textit{and} \\ \boldsymbol{\Delta}_3 &= \mathbf{U}_0 + (u-1)\,\mathbf{U}_1 + u\,(v-1)\,\mathbf{W} = (\mathbf{U}_0 - \mathbf{U}_1) + u\,(\mathbf{U}_1 - \mathbf{W}) + uv\mathbf{W}. \end{aligned}$$

Corollary

If $\mathbf{W} = \mathbf{0}$, *then*

$$
\begin{aligned}
\boldsymbol{\Delta}_1 &= \mathbf{U}_0 - \mathbf{U}_1, \\
\boldsymbol{\Delta}_2 &= \mathbf{U}_0 + (u-1)\,\mathbf{U}_1 = (\mathbf{U}_0 - \mathbf{U}_1) + u\mathbf{U}_1, \quad \textit{and} \\
\boldsymbol{\Delta}_3 &= \mathbf{U}_0 + (u-1)\,\mathbf{U}_1 = (\mathbf{U}_0 - \mathbf{U}_1) + u\mathbf{U}_1.
\end{aligned}
$$

Thus, we see that $\boldsymbol{\Delta}_3 = \boldsymbol{\Delta}_2$.

Covariance Structure

General Setup

$$\boldsymbol{V} = \sum_{j=1}^{\kappa} \boldsymbol{Q}_j \otimes \boldsymbol{\Gamma}_j \quad = \quad \boldsymbol{Q}_1 \otimes \boldsymbol{\Gamma}_1 + \cdots + \boldsymbol{Q}_\kappa \otimes \boldsymbol{\Gamma}_\kappa,$$

where the matrices $\boldsymbol{Q}_j$ are are symmetric and idempotent and

$$\boldsymbol{Q}_j \boldsymbol{Q}_l \quad = \quad \boldsymbol{0}, \;\; j \neq l, j, l = 1, \ldots, \kappa,$$

and $$\sum_{j=1}^{\kappa} \boldsymbol{Q}_j \quad = \quad \boldsymbol{I}_{uv}.$$

Covariance Structure

$$\boldsymbol{V}_{\kappa} = \sum_{i=1}^{\kappa} \left(\bigotimes_{j=i+2}^{\kappa} \boldsymbol{I}_{a_{\kappa-j+i+2}} \right) \otimes \boldsymbol{K}_{a_{i+1}} \otimes \left(\bigotimes_{l=1}^{i} \left(\frac{1}{a_{i-l+1}} \boldsymbol{J}_{a_{i-l+1}} \right) \right) \otimes \boldsymbol{\Gamma}_i^{(\kappa)}$$

Parametrization

$$\boldsymbol{\Gamma}_i^{(\kappa)} = \sum_{j=1}^{i} (a_j - 1) b_{j-1} \boldsymbol{U}_j - b_j \boldsymbol{U}_{i+1}, i = 1, \ldots, \kappa - 1.$$

$$\boldsymbol{Q}_i = \left(\bigotimes_{j=i+2}^{\kappa} \boldsymbol{I}_{a_{\kappa-j+i+2}} \right) \otimes \boldsymbol{K}_{a_{i+1}} \otimes \left(\bigotimes_{l=1}^{i} \left(\frac{1}{a_{i-l+1}} \boldsymbol{J}_{a_{i-l+1}} \right) \right)$$

Lemma

Let $\boldsymbol{V}$ be a $(\kappa)-$exchangeable covariance structure.

1. *If $\boldsymbol{\Gamma}_j$, $j = 1, \ldots, \kappa$ are non singular matrices, the matrix $\boldsymbol{V}$ is non singular, and its inverse is given by*

$$\boldsymbol{V}^{-1} = \sum_{j=1}^{\kappa} \boldsymbol{Q}_j \otimes \boldsymbol{\Gamma}_j^{-1}. \tag{1}$$

2. *The determinant of $\boldsymbol{V}$ is given by*

$$|\boldsymbol{V}| = \prod_{j=1}^{\kappa} |\boldsymbol{\Gamma}_j|^{g_j}, \tag{2}$$

The quantities g_j are rank of $\boldsymbol{Q}_j$.

Auxiliary Results

- $\underset{m\times 1}{\boldsymbol{Z}_{t1}}$ and $\underset{m\times(u-1)}{\boldsymbol{Z}_{t2}}$, $t = 1, \ldots, v$ represent the m response variables at the first and the rest of $u-1$ locations respectively at the t^{th} time point in the doubly transformed domain
- $\underset{r-1\times 1}{\boldsymbol{U}_{t1}}$ and $\underset{r-1\times u-1}{\boldsymbol{U}_{t2}}$ are the $r-1$ values of predictor variables at the first and the rest of $u-1$ locations respectively at the t^{th} time point in the doubly transformed domain

Theorem

Let $\boldsymbol{Y}$ *represent the random matrix of three-level multivariate response variables. If*

$$\boldsymbol{\Gamma}^{\bullet}\boldsymbol{\Gamma}(Vec \underset{m\times(u)v}{\boldsymbol{Y}}) = \begin{bmatrix} Vec(\underset{m\times 1}{\boldsymbol{Z}_{11}} : \underset{m\times(u-1)}{\boldsymbol{Z}_{12}}) \\ Vec(\underset{m\times 1}{\boldsymbol{Z}_{21}} : \underset{m\times(u-1)}{\boldsymbol{Z}_{22}}) \\ \vdots \\ Vec(\underset{m\times 1}{\boldsymbol{Z}_{v1)}} : \underset{m\times(u-1)}{\boldsymbol{Z}_{v2}}) \end{bmatrix},$$

then all the components $\boldsymbol{Z}_{11}, \boldsymbol{Z}_{12}, \boldsymbol{Z}_{21}, \ldots, \boldsymbol{Z}_{v2}$ *are independently normally distributed such that*

$$\begin{aligned} & \boldsymbol{Z}_{11} \sim N_{m,1}(\sqrt{uv}\boldsymbol{\alpha} + \boldsymbol{\gamma}'\boldsymbol{U}_{11}, \boldsymbol{\Delta}_3, 1), \\ & \boldsymbol{Z}_{t1} \sim N_{m,1}(\boldsymbol{\gamma}'\boldsymbol{U}_{t1}, \boldsymbol{\Delta}_2, 1) \;\; t = 2, 3, \ldots, v, \\ \text{and} \quad & \boldsymbol{Z}_{t2} \sim N_{m,u-1}(\boldsymbol{\gamma}'\boldsymbol{U}_{t2}, \boldsymbol{\Delta}_1, \boldsymbol{I}_{u-1}) \;\; t = 1, 2, \ldots, v. \end{aligned}$$

Corollary

If $\boldsymbol{W} = \boldsymbol{0}$, *then*

$$\begin{aligned} & \boldsymbol{Z}_{11} \sim N_{m,1}(\sqrt{uv}\boldsymbol{\alpha} + \gamma' \boldsymbol{U}_{11}, \boldsymbol{\Delta}_2, 1), \\ & \boldsymbol{Z}_{t1} \sim N_{m,1}(\gamma' \boldsymbol{U}_{t1}, \boldsymbol{\Delta}_2, 1) \;\; t = 2, 3, \ldots, v, \\ \textit{and} \quad & \boldsymbol{Z}_{t2} \sim N_{m,u-1}(\gamma' \boldsymbol{U}_{t2}, \boldsymbol{\Delta}_1, \boldsymbol{I}_{u-1}) \;\; t = 1, 2, \ldots, v. \end{aligned}$$

Corollary

If $v = 1$, *the DEGLM reduces to EGLM. For* $v = 1$, *we have*

$$\begin{aligned} & \boldsymbol{Z}_{11} \sim N_{m,1}(\sqrt{u}\boldsymbol{\alpha} + \gamma' \boldsymbol{U}_{11}, \boldsymbol{U}_0 - (u-1)\boldsymbol{U}_1, 1), \\ \textit{and} \quad & \boldsymbol{Z}_{12} \sim N_{m,u-1}(\gamma' \boldsymbol{U}_{12}, \boldsymbol{U}_0 - \boldsymbol{U}_1, \boldsymbol{I}_{u-1}). \end{aligned}$$

Fixed Effects

Estimators

$$\widehat{\boldsymbol{\alpha}}' = (uv)^{-1/2}(\boldsymbol{Z}'_{11} - \boldsymbol{U}'_{11}\widehat{\boldsymbol{\gamma}})$$

$$\widehat{\boldsymbol{\gamma}} = \Big(\sum_{j=1}^{2}\sum_{t=1}^{v}\boldsymbol{U}_{tj}\boldsymbol{U}'_{tj} - \boldsymbol{U}_{11}\boldsymbol{U}'_{11}\Big)^{-1}\Big(\sum_{j=1}^{2}\sum_{t=1}^{v}\boldsymbol{U}_{tj}\boldsymbol{Z}'_{tj} - \boldsymbol{U}_{11}\boldsymbol{Z}'_{11}\Big)$$

Fixed Effects

Estimators: $v = 1$

$$\begin{aligned}
\widehat{\alpha}' &= (u)^{-1/2}\left(\boldsymbol{Z}'_{11} - \boldsymbol{U}'_{11}\widehat{\gamma}\right), \\
\widehat{\gamma} &= \left(\sum_{j=1}^{2} \boldsymbol{U}_{1j}\boldsymbol{U}'_{1j} - \boldsymbol{U}_{11}\boldsymbol{U}'_{11}\right)^{-1}\left(\sum_{j=1}^{2} \boldsymbol{U}_{1j}\boldsymbol{Z}'_{1j} - \boldsymbol{U}_{11}\boldsymbol{Z}'_{11}\right) \\
&= \left(\boldsymbol{U}_{12}\boldsymbol{U}'_{12}\right)^{-1}\left(\underset{r-1\times u-1}{\boldsymbol{U}_{12}}\boldsymbol{Z}'_{12}\right),
\end{aligned}$$

Fixed Effects

Estimators: $v = 1$

$$
\begin{aligned}
\widehat{\boldsymbol{\alpha}}' &\sim N_{1,m}\Big(\boldsymbol{\alpha}', 1, \\
&\frac{1}{uv}\Big[\boldsymbol{\Delta}_3 + \boldsymbol{U}'_{11}\boldsymbol{A}^{-1}\sum_{t=2}^{v}\left[\boldsymbol{U}_{t1}\boldsymbol{U}'_{t1}\right]\boldsymbol{A}^{-1}\boldsymbol{U}_{11}\boldsymbol{\Delta}_2 \\
&+\boldsymbol{U}'_{11}\boldsymbol{A}^{-1}\sum_{t=1}^{v}\left[\boldsymbol{U}_{t2}\boldsymbol{U}'_{t2}\right]\boldsymbol{A}^{-1}\boldsymbol{U}_{11}\boldsymbol{\Delta}_1\Big]\Big), \\
\mathrm{Vec}\widehat{\boldsymbol{\gamma}'} &\sim N_{(r-1)m}\Big(\mathrm{Vec}\boldsymbol{\gamma}', \\
&\boldsymbol{A}^{-1}\sum_{t=1}^{v}\left[\boldsymbol{U}_{t1}\boldsymbol{U}'_{t1}\right]\boldsymbol{A}^{-1}\otimes\boldsymbol{\Delta}_2 + \boldsymbol{A}^{-1}\sum_{t=1}^{v}\left[\boldsymbol{U}_{t2}\boldsymbol{U}'_{t2}\right]\boldsymbol{A}^{-1}\otimes\boldsymbol{\Delta}_1\Big),
\end{aligned}
$$

where $\boldsymbol{A} = \sum_{j=1}^{2}\sum_{t=1}^{v}\boldsymbol{U}_{tj}\boldsymbol{U}'_{tj} - \boldsymbol{U}_{11}\boldsymbol{U}'_{11}$

Fixed Effects

Estimator Distribution: $\mathbf{W} = \mathbf{0}$

$$
\begin{aligned}
\widehat{\boldsymbol{\alpha}}' &\sim N_{1,m}\Big(\boldsymbol{\alpha}', 1, \\
&\frac{1}{uv}\Big[\boldsymbol{\Delta}_2 + \boldsymbol{U}_{11}'\boldsymbol{A}^{-1}\sum_{t=2}^{v}\left[\boldsymbol{U}_{t1}\boldsymbol{U}_{t1}'\right]\boldsymbol{A}^{-1}\boldsymbol{U}_{11}\boldsymbol{\Delta}_2 \\
&+\boldsymbol{U}_{11}'\boldsymbol{A}^{-1}\sum_{t=1}^{v}\left[\boldsymbol{U}_{t2}\boldsymbol{U}_{t2}'\right]\boldsymbol{A}^{-1}\boldsymbol{U}_{11}\boldsymbol{\Delta}_1\Big]\Big), \\
\widehat{\boldsymbol{\gamma}} &= \boldsymbol{A}^{-1}\Big(\sum_{t=1}^{v}\boldsymbol{U}_{t1}\boldsymbol{Z}_{t1}' + \sum_{t=1}^{v}\boldsymbol{U}_{t2}\boldsymbol{Z}_{t2}'\Big) \sim N_{(r-1),m}\Big(\boldsymbol{\gamma}, \\
&\boldsymbol{A}^{-1}\sum_{i=2}^{v}\left[\boldsymbol{U}_{t1}\boldsymbol{U}_{t1}'\right]\boldsymbol{A}^{-1}\otimes\boldsymbol{\Delta}_2 + \boldsymbol{A}^{-1}\sum_{t=1}^{v}\left[\boldsymbol{U}_{t2}\boldsymbol{U}_{t2}'\right]\boldsymbol{A}^{-1}\otimes\boldsymbol{\Delta}_1\Big).
\end{aligned}
$$

where $\boldsymbol{A} = \sum_{j=1}^{2}\sum_{t=1}^{v}\boldsymbol{U}_{tj}\boldsymbol{U}_{tj}' - \boldsymbol{U}_{11}\boldsymbol{U}_{11}'$

Fixed Effects

Estimator Distribution: $\mathbf{W} = \mathbf{0}$

$$\begin{aligned}
\widehat{\alpha}' &\sim N_{1,m}\left(\alpha', 1, \frac{1}{u}\left[\boldsymbol{\Delta}_3 + \boldsymbol{U}_{11}'\boldsymbol{A}^{-1}(\boldsymbol{U}_{12}\boldsymbol{U}_{12}')\boldsymbol{A}^{-1}\boldsymbol{U}_{11}\boldsymbol{\Delta}_1\right]\right)\\
&\sim N_{1,m}\left(\alpha', 1, +\frac{1}{u}\left(1 + \boldsymbol{U}_{11}'(\boldsymbol{U}_{12}\boldsymbol{U}_{12}')^{-1}\boldsymbol{U}_{11}\right)(\boldsymbol{U}_0 - \boldsymbol{U}_1)\right)\\
\mathrm{Vec}\widehat{\gamma}' &\sim N_{(r-1)m}\mathrm{Vec}\left(\gamma', (\boldsymbol{U}_{12}\boldsymbol{U}_{12}')^{-1}\boldsymbol{U}_{12}\boldsymbol{U}_{12}'(\boldsymbol{U}_{12}\boldsymbol{U}_{12}')^{-1} \otimes \boldsymbol{\Delta}_1\right)\\
&\sim N_{(r-1)m}\left(\mathrm{Vec}\gamma', (\boldsymbol{U}_{12}\boldsymbol{U}_{12}')^{-1} \otimes \boldsymbol{\Delta}_1\right),\\
\widehat{\gamma} &\sim N_{(r-1),m}\left(\gamma, (\boldsymbol{U}_{12}\boldsymbol{U}_{12}')^{-1}, (\boldsymbol{U}_0 - \boldsymbol{U}_1)\right)
\end{aligned}$$

Covariance Components

Estimators

Estimates for $\widehat{\boldsymbol{\Delta}}_1$ and $\widehat{\boldsymbol{\Delta}}_2$

- $\widehat{\boldsymbol{\Delta}}_1 = \dfrac{1}{v(u-r)} \boldsymbol{S}_1$
- $\widehat{\boldsymbol{\Delta}}_2 = \dfrac{1}{(v-r)} \boldsymbol{S}_2$

with

- $\boldsymbol{S}_1 = \sum_{t=1}^{v} \boldsymbol{S}_{t2} = \sum_{t=2}^{v} \left[\boldsymbol{Z}_{t1} - (\boldsymbol{U}'_{t1}\widehat{\gamma})'\right]\left[\boldsymbol{Z}'_{t1} - (\boldsymbol{U}'_{t1}\widehat{\gamma})\right]$
- $\boldsymbol{S}_2 = \sum_{t=2}^{v} \boldsymbol{S}_{t1} = \sum_{t=2}^{v} \left[\boldsymbol{Z}_{t2} - (\boldsymbol{U}'_{t2}\widehat{\gamma})'\right]\left[\boldsymbol{Z}'_{t2} - (\boldsymbol{U}'_{t2}\widehat{\gamma})\right]$

Covariance Components

Estimators Distribution

$$\sum_{t=2}^{v} \boldsymbol{S}_{t1} \sim W_m\Big(\boldsymbol{\Delta}_2, (v-r)\Big)$$
$$\boldsymbol{S}_{t2} \sim W_m\Big(\boldsymbol{\Delta}_1, u-r\Big)$$

Distributions

- $\boldsymbol{S}_1 \sim W_m\Big(\boldsymbol{\Delta}_1, v(u-r)\Big)$
- $\boldsymbol{S}_2 \sim W_m\Big(\boldsymbol{\Delta}_2, (v-r)\Big)$

Maximum Likelihood

Log-likelihood

$$\ell(\boldsymbol{y}) = \log\left(|\boldsymbol{V}|\right) + \sum_{j=1}^{\kappa} (\boldsymbol{y} - (\boldsymbol{X} \otimes \boldsymbol{I}_p)\beta)'(\boldsymbol{Q}_j \otimes \boldsymbol{\Gamma}_j^{-1})\,(\boldsymbol{y} - (\boldsymbol{X} \otimes \boldsymbol{I}_p)\beta)$$

$$= \underbrace{\sum_{j=1}^{\kappa} g_j \log\left(|\boldsymbol{\Gamma}_j|\right) + \left(\boldsymbol{y} - (\boldsymbol{X} \otimes \boldsymbol{I}_p)\hat{\boldsymbol{\beta}}\right)' \left(\sum_{j=1}^{\kappa} \boldsymbol{Q}_j \otimes \boldsymbol{\Gamma}_j^{-1}\right) \left(\boldsymbol{y} - (\boldsymbol{X} \otimes \boldsymbol{I}_p)\hat{\boldsymbol{\beta}}\right)}_{\text{random effects}}$$

$$+ \underbrace{(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})' \left(\sum_{j=1}^{\kappa} (\boldsymbol{X}'\boldsymbol{Q}_j\boldsymbol{X}) \otimes \boldsymbol{\Gamma}_j^{-1}\right) (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})}_{\text{fixed effects}}$$

Estimators

Fixed Effects

$$\underset{pm\times 1}{\hat{\boldsymbol{\beta}}} = \left((\underset{m\times r}{\boldsymbol{X}'}\underset{r\times m}{\boldsymbol{X}})^{-1}\boldsymbol{X}' \otimes \boldsymbol{I}_p \right) \underset{pr\times 1}{\boldsymbol{y}}$$

$$\text{Vec}(\underset{p\times m}{\hat{\boldsymbol{B}}}) = \text{Vec}(\underset{p\times r}{\boldsymbol{Y}}(\boldsymbol{X}(\underset{m\times r}{\boldsymbol{X}'}\underset{r\times m}{\boldsymbol{X}})^{-1})))$$

$$\underset{p\times m}{\hat{\boldsymbol{B}}} = \underset{p\times r}{\boldsymbol{Y}}(\boldsymbol{X}(\underset{m\times r}{\boldsymbol{X}'}\underset{r\times m}{\boldsymbol{X}})^{-1}))$$

Theorem

The $h_j p-$dimensional random variable $\boldsymbol{z}_j = \boldsymbol{B}_j'\boldsymbol{y}$ for $j = 1, \ldots, \kappa$, is normally distributed distributed with mean vector $\mathbf{0}$ and variance-covariance matrix $(\boldsymbol{I}_{h_j} \otimes \boldsymbol{\Gamma}_j)$, with $\mathbf{B}_j\mathbf{B}_j' = \mathbf{I} - \mathbf{P}_{\mathbf{X}}\mathbf{Q}_j$ and $h_j = \mathrm{rank}(\boldsymbol{B}_j)$.

Estimators

Covariance Components

$$\hat{\mathbf{\Gamma}}_j = \frac{1}{g_j}\mathbf{Z}_j'\mathbf{Z}_j, \;\; j = 1, \ldots, \kappa.$$

Note that $\frac{g_j}{h_j}\hat{\mathbf{\Gamma}}_j$ is an unbiased estimator of $\mathbf{\Gamma}_j$.

Estimators Distributions

- $\hat{\beta} \sim \mathcal{N}(\beta, \sum_{j=1}^{\kappa} \left((\boldsymbol{X}'\boldsymbol{X})^{-1} \boldsymbol{X}' \boldsymbol{Q}_j \boldsymbol{X} (\boldsymbol{X}'\boldsymbol{X})^{-1} \otimes \boldsymbol{\Gamma}_j \right)$
- $g_j \hat{\boldsymbol{\Gamma}}_j \sim W_{h_j}(\boldsymbol{\Gamma}_j)$

Parametric Bootstrap

Definition

Let x_0 be the observed value of the random variable x with density $f(x_0;\zeta)$. $T = T(x;x_0,\zeta)$ is said to be a generalized test statistic if the following three properties hold:

- For fixed x_0 and $\zeta = (\theta_0, \eta)$, the distribution of $T(x;x_0,\zeta)$ is independent of nuisance parameter η.
- $t_{\text{obs}} = T(x;x_0,\zeta)$ does not depend on unknown parameters.
- For fixed x_0 and η ,$P\big[T(x;x_0,\zeta) \geq t\big]$ is either stochastically increasing or decreasing in θ for any given t.

The generalized p-value for hypothesis $H_0 : \theta \leq \theta_0$ vs. $H_1 : \theta > \theta_0$ is then defined by

$$p = \sup_{\theta \leq \theta_0} P\big[T(x;x_0,\theta,\eta) \geq t\big] = P\big[T(x;x_0,\theta_0,\eta) \geq t\big].$$

Parametric Bootstrap

Auxiliary Results

$$\begin{aligned} \boldsymbol{Z}_{\bullet 1} &= [\boldsymbol{Z}_{21} : \cdots : \boldsymbol{Z}_{v1}] \\ \boldsymbol{Z}_{\bullet 2} &= [\boldsymbol{Z}_{12} : \cdots : \boldsymbol{Z}_{v2}] \end{aligned}.$$

$$\begin{aligned} \boldsymbol{Z}_{\bullet 1} &\sim N(\gamma' \boldsymbol{U}_{\bullet 1}, \boldsymbol{\Delta}_2, \boldsymbol{I}_{v-1}) \\ \boldsymbol{Z}_{\bullet 2} &\sim N(\gamma' \boldsymbol{U}_{\bullet 2}, \boldsymbol{\Delta}_1, \boldsymbol{I}_{v(u-1)}), \end{aligned}$$

with

$$\begin{aligned} \boldsymbol{U}_{\bullet 1} &= [\boldsymbol{U}_{21} : \cdots : \boldsymbol{U}_{v1}] \\ \boldsymbol{U}_{\bullet 2} &= [\boldsymbol{U}_{12} : \cdots : \boldsymbol{U}_{v2}]. \end{aligned}$$

Parametric Bootstrap

Pivotal Quantities

$$\boldsymbol{D}_i = \widehat{\boldsymbol{\Omega}}_i \boldsymbol{T}_i^{-1} (\boldsymbol{T}_i^{-1})' \widehat{\boldsymbol{\Omega}}_i'$$

is a pivotal quantity for $\boldsymbol{\Delta}_i$, $i = 1, 2$

- $\boldsymbol{\Omega}_i^{-1} \boldsymbol{S}_i (\boldsymbol{\Omega}_i')^{-1} \sim \boldsymbol{T}_i \boldsymbol{T}_i'$
  - $\boldsymbol{R}_i^{-1} \widehat{\boldsymbol{\Omega}}_i \sim \boldsymbol{T}_i$
- $\boldsymbol{\Omega}_i$ is the lower triangular matrix on the Cholesky decomposition of $\boldsymbol{\Delta}_i$ for $i = 1, 2$
- $\boldsymbol{R}_i$ as the lower triangular matrix in the Cholesky decomposition of $\widehat{\boldsymbol{\Delta}}_i$ for $i = 1, 2$
- $\boldsymbol{T}_i$ is a lower triangular matrix, deriving from the Bartlett decomposition of the Wishart distribution $\boldsymbol{S}_i$ $i = 1, 2$

Parametric Bootstrap

Slope Parameter γ

$$T_\gamma = \frac{(\text{Vec}\widehat{\gamma} - \text{Vec}\gamma_0)' \boldsymbol{V}_\gamma^{-1} (\text{Vec}\widehat{\gamma} - \text{Vec}\gamma_0)}{U_{r-1}},$$

with

$$\boldsymbol{V}_\gamma = \boldsymbol{B} \otimes \boldsymbol{D}_1 + \boldsymbol{C} \otimes \boldsymbol{D}_2,$$

where

- $\boldsymbol{B} = \boldsymbol{A}^{-1} \sum_{t=1}^{v} \left[\boldsymbol{U}_{t2} \boldsymbol{U}_{t2}' \right] \boldsymbol{A}^{-1}$
- $\boldsymbol{C} = \boldsymbol{A}^{-1} \sum_{t=1}^{v} \left[\boldsymbol{U}_{t1} \boldsymbol{U}_{t1}' \right] \boldsymbol{A}^{-1}$
- $U_{r-1} \sim \chi^2_{r-1}$

Parametric Bootstrap

Slope Parameter γ

Under the null hypothesis $H_0 : \text{Vec}\boldsymbol{\gamma} = \mathbf{0}$, which is equivalent to

$$H_0 : H_0 : \text{Vec}\boldsymbol{\gamma} = \mathbf{0} \Leftrightarrow (\text{Vec}\boldsymbol{\gamma})'\text{Vec}\boldsymbol{\gamma} = \mathbf{0},$$

the variable

Generalized Test Variable

$$T_{\gamma} = \frac{(\text{vec}\widehat{\boldsymbol{\gamma}})' \boldsymbol{V}_{\gamma}^{-1} \text{vec}\widehat{\boldsymbol{\gamma}}}{U_{r-1}}$$

Parametric Bootstrap

Intercept parameter $\boldsymbol{\alpha}$

$$H_0 = \boldsymbol{\alpha} = \mathbf{0} \Leftrightarrow \boldsymbol{\alpha}'\boldsymbol{\alpha} = 0$$

Generalized Test Variable

$$T_{\boldsymbol{\alpha}} = \frac{(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha})' \boldsymbol{V}_{\boldsymbol{\alpha}}^{-1} (\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha})}{U_m}$$

with

$$\boldsymbol{V}_{\boldsymbol{\alpha}} = \frac{1}{uv}\left(\boldsymbol{U}_{11}\boldsymbol{B}\boldsymbol{U}_{11}'\boldsymbol{D}_1 + (\boldsymbol{I} + \boldsymbol{U}_{11}\boldsymbol{C}\boldsymbol{U}_{11}')\boldsymbol{D}_2\right),$$

where $U_m \sim \chi^2_m$

Multivariate Satterthwaite Approximation

Slope Parameter γ

$$\boldsymbol{Z}_{\bullet 1} = [\boldsymbol{Z}_{21} : \cdots : \boldsymbol{Z}_{v1}]$$
$$\boldsymbol{Z}_{\bullet 2} = [\boldsymbol{Z}_{12} : \cdots : \boldsymbol{Z}_{v2}],$$

under the null hypothesis $H_0 : \gamma = \mathbf{0}$,

$$\boldsymbol{Z}_{\bullet 1}\boldsymbol{Z}'_{\bullet 1} \sim W_m(\boldsymbol{\Delta}_2, v-1)$$
$$\boldsymbol{Z}_{\bullet 2}\boldsymbol{Z}'_{\bullet 2} \sim W_m(\boldsymbol{\Delta}_1, v(u-1)),$$

Multivariate Satterthwaite Approximation

Slope Parameter γ

$$\boldsymbol{S}_\gamma = \boldsymbol{Z}_{\bullet 1}\boldsymbol{Z}'_{\bullet 1} + \boldsymbol{Z}_{\bullet 2}\boldsymbol{Z}'_{\bullet 2}$$

has an approximate Wishart distribution with parameters

$$\boldsymbol{\Delta} = \frac{1}{g}\left(\boldsymbol{\Delta}_1 + \boldsymbol{\Delta}_2\right)$$

$$g = \left(\frac{|\boldsymbol{\Delta}_1 + \boldsymbol{\Delta}_2|^{m+1}}{|\boldsymbol{V}_\gamma|}\right)^{\frac{1}{m^*}},$$

where

$$\boldsymbol{V}_\gamma = \boldsymbol{K}\left(\frac{1}{v(u-1)}\boldsymbol{\Delta}_1 \otimes \boldsymbol{\Delta}_1 + \frac{1}{v-1}\boldsymbol{\Delta}_2 \otimes \boldsymbol{\Delta}_2\right)\boldsymbol{K}^+.$$

$m^* = \frac{m(m+1)}{2}$ and $\boldsymbol{K}$ is a $m^2 \times m^*$, with generic element $[k]_{ij,gh} = \frac{1}{2}(\delta_{ig}\delta_{jh} + \delta_{ih}\delta_{jg})$, $i, j \leq m$, $g \leq h \leq m$, $\delta_{a,b}$ is the Kronecker delta, with lexicographical ordering on the indexes

Multivariate Satterthwaite Approximation

Intercept Parameter α

$$\boldsymbol{Y}_{\alpha} = \boldsymbol{Z}_{11} - \gamma' \boldsymbol{U}_{11} \sim N(\sqrt{uv}\alpha, \boldsymbol{\Lambda}),$$

with

$$\boldsymbol{\Lambda} = \boldsymbol{U}_{11}' \boldsymbol{B} \boldsymbol{U}_{11} \boldsymbol{\Delta}_1 + (1 + \boldsymbol{U}_{11}' \boldsymbol{C} \boldsymbol{U}_{11}) \boldsymbol{\Delta}_2,$$

$$c_1 = \boldsymbol{U}_{11}' \boldsymbol{B} \boldsymbol{U}_{11}$$
$$c_2 = 1 + \boldsymbol{U}_{11}' \boldsymbol{C} \boldsymbol{U}_{11}{}^{,}$$

Multivariate Satterthwaite Approximation

Intercept Parameter α

$$\boldsymbol{S}_{\alpha} = c_1 \boldsymbol{S}_1 + c_2 \boldsymbol{S}_2.$$

is approximately distributed as $W_m(\frac{1}{h}\boldsymbol{\Lambda}, h)$, where

$$h = \left(\frac{|\boldsymbol{\Lambda}|^{m+1}}{|\boldsymbol{V}_{\alpha}|} \right),$$

$$\boldsymbol{V}_{\alpha} = \boldsymbol{K} \left(\frac{c_1^2}{v(u-1)} \boldsymbol{\Delta}_1 \otimes \boldsymbol{\Delta}_1 + \frac{c_2^2}{v-1} \boldsymbol{\Delta}_2 \otimes \boldsymbol{\Delta}_2 \right) \boldsymbol{K}^{+},$$

Test Variable

$\boldsymbol{Y}'_{\alpha} \boldsymbol{S}_{\alpha}^{-1} \boldsymbol{Y}_{\alpha}$, under the null hypothesis $H_0 : \alpha = \boldsymbol{0}$, has an approximate $F_{m,h}$ distribution.

Hypothesis Formulation

$$\boldsymbol{C}'\boldsymbol{B} = \boldsymbol{B}_* \Leftrightarrow \boldsymbol{C}'(\boldsymbol{B} - \boldsymbol{B}_0) = \boldsymbol{0},$$

$$(\boldsymbol{C} \otimes \boldsymbol{I}_p)'(\beta - \beta_0) = \boldsymbol{0}$$

Admit that $\boldsymbol{C} = \boldsymbol{X}'\boldsymbol{D}$ and, consequently, that $\boldsymbol{D} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{C}$

$$\left(\boldsymbol{D}'\boldsymbol{X} \otimes \boldsymbol{I}_p\right)(\beta - \beta_0) = \boldsymbol{0}$$

- $\beta - \beta_0$ is in the kernel of $\boldsymbol{D}'\boldsymbol{X} \otimes \boldsymbol{I}_p = \boldsymbol{C}' \otimes \boldsymbol{I}_p$, or equivalently, in the kernel of $(\boldsymbol{C}'\boldsymbol{C}) \otimes \boldsymbol{I}_p$
- $\boldsymbol{X}(\beta - \beta_0)$ is in the kernel of $\boldsymbol{D}' \otimes \boldsymbol{I}_p$ or $\boldsymbol{D}'\boldsymbol{D} \otimes \boldsymbol{I}_p$

Log-likelihood under H_0

- $\boldsymbol{P}_{\boldsymbol{D}'}$ is the orthogonal projection matrix on the $\ker(\boldsymbol{D}')$
- $\boldsymbol{P}^{\perp}_{\boldsymbol{D}'}$ is the orthogonal projection matrix on the orthogonal complement of $\ker(\boldsymbol{D}')$, with $\dim\big(\ker(\boldsymbol{D}')\big) = c$
- $\boldsymbol{B}_{\boldsymbol{D}'}$ and $\boldsymbol{B}^{\perp}_{\boldsymbol{D}'}$ be matrices whose columns are orthonormal basis for $\ker(\boldsymbol{D}')$ and $\ker(\boldsymbol{D}')^{\perp}$, respectively

$$\ell = \sum_{j=1}^{\kappa} g_j \log\left(|\boldsymbol{\Gamma}_j|\right) + \left(\boldsymbol{y}_0 - ((\boldsymbol{P}^{\perp}_{\boldsymbol{D}'}\boldsymbol{X}) \otimes \boldsymbol{I}_p)(\boldsymbol{\beta} - \boldsymbol{\beta}_0)\right)' \left(\sum_{j=1}^{\kappa} \boldsymbol{Q}_j \otimes \boldsymbol{\Gamma}_j^{-1}\right) \left(\boldsymbol{y}_0 - ((\boldsymbol{P}^{\perp}_{\boldsymbol{D}'}\boldsymbol{X}) \otimes \boldsymbol{I}_p)(\boldsymbol{\beta} - \boldsymbol{\beta}_0)\right),$$

$$\boldsymbol{y}_0 = \boldsymbol{y} - ((\boldsymbol{P}^{\perp}_{\boldsymbol{D}'}\boldsymbol{X}) \otimes \boldsymbol{I}_p)\boldsymbol{\beta}_0,\ c_j = \mathrm{rank}((\boldsymbol{I} - \boldsymbol{P}_0)\boldsymbol{Q}_j) \geq \mathrm{rank}((\boldsymbol{I} - \boldsymbol{P}_{\boldsymbol{X}})\boldsymbol{Q}_j) = h_j.$$

Hypothesis Test

Estimators under H_0

$$\hat{\boldsymbol{B}}_0 = \boldsymbol{Y}\boldsymbol{P}^{\perp}_{\boldsymbol{D}'}\boldsymbol{X}'(\boldsymbol{X}'\boldsymbol{P}^{\perp}_{\boldsymbol{D}'}\boldsymbol{X})^{-}$$

$$\hat{\boldsymbol{\Gamma}}_{0,j} = \frac{1}{g_j}\boldsymbol{Y}'(\boldsymbol{I} - \boldsymbol{P}_0)\boldsymbol{Q}_j\boldsymbol{Y}; j = 1, \ldots, \kappa$$

Test Variable

$$LRT = \frac{\prod_{j=1}^{\kappa} |\hat{\boldsymbol{\Gamma}}_{0,j}|^{\frac{g_j}{2}}}{\prod_{j=1}^{\kappa} |\hat{\boldsymbol{\Gamma}}_j|^{\frac{g_j}{2}}}$$

Hypothesis Test

Special Case

$$H_0 : \boldsymbol{BC} = 0$$

- $\boldsymbol{B}_* = \mathbf{0}$
- $\boldsymbol{C} = [1 \quad 0 \quad \dots \quad 0]'$
- $\boldsymbol{P}_{\boldsymbol{D}'} = \boldsymbol{I} - \frac{1}{n}\boldsymbol{J}_n$

Test Variable

$$LRT = \frac{\left|\hat{\boldsymbol{\Gamma}}_{0,1}\right|^{\frac{g_1}{2}}}{\left|\hat{\boldsymbol{\Gamma}}_1\right|^{\frac{g_1}{2}}}$$

Acknowledgements

This work was funded by project PEst-OE/MAT/UI0297/2014.

THANK YOU FOR YOUR ATTENTION!