# Hypothesis testing in multilevel models with block circular covariance structures

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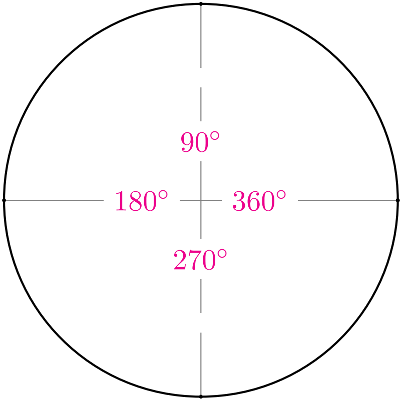
# Circular dependence: Circular Toeplitz (CT) matrix




$$\begin{pmatrix} \tau_0 & \tau_1 & \tau_2 & \tau_1 \\ \tau_1 & \tau_0 & \tau_1 & \tau_2 \\ \tau_2 & \tau_1 & \tau_0 & \tau_1 \\ \tau_1 & \tau_2 & \tau_1 & \tau_0 \end{pmatrix}$$

# CT matrix, cont.

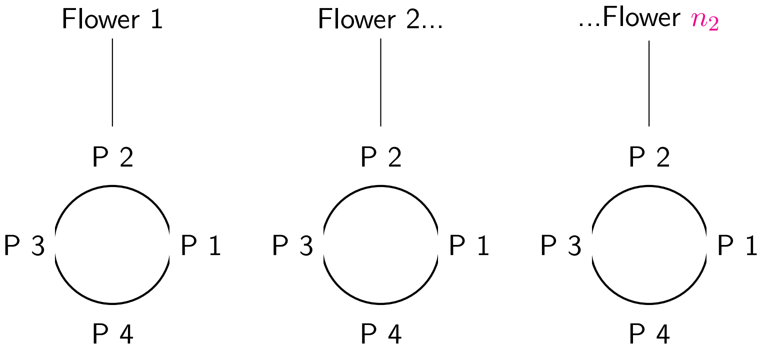
An $n \times n$ matrix $\boldsymbol{T}$ of the form

$$\boldsymbol{T} = \begin{pmatrix} t_0 & t_1 & t_2 & \cdots & t_1 \\ t_1 & t_0 & t_1 & \cdots & t_2 \\ t_2 & t_1 & t_0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ t_1 & t_2 & \cdots & t_1 & t_0 \end{pmatrix} = Toep(t_0, t_1, t_2 \ldots, t_1)$$

is called a *symmetric circular Toeplitz matrix*. The matrix $\boldsymbol{T} = (t_{ij})$ depends on $[n/2] + 1$ parameters, where $[.]$ stands for the integer part, and for $i, j = 1, \ldots, n$,

$$t_{ij} = \begin{cases} t_{|j-i|} & |j-i| \leqslant [n/2], \\ t_{n-|j-i|} & \text{otherwise.} \end{cases}$$

# A specific structure



# Outline

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# Balanced three-level model

- $\mathbf{y}_k = \mu \mathbf{1}_p + \boldsymbol{Z}_3 \boldsymbol{\alpha} + \boldsymbol{Z}_2 \boldsymbol{\beta} + \boldsymbol{\epsilon}_k$, $k = 1, \ldots, n$,
  where $p = n_2 n_1$, $\boldsymbol{Z}_3 = \boldsymbol{I}_{n_2} \otimes \mathbf{1}_{n_1}$ and $\boldsymbol{Z}_2 = \boldsymbol{I}_{n_2} \otimes \boldsymbol{I}_{n_1}$,
  $Cov(\boldsymbol{\alpha}) = \boldsymbol{V}_3 \geqslant 0$, $Cov(\boldsymbol{\beta}) = \boldsymbol{V}_2 \geqslant 0$ and
  $Var(\boldsymbol{\epsilon}_k) = \sigma^2 \boldsymbol{I}_p > 0$, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are independent.
- $\mathbf{y}_k \sim N_p(\mu \mathbf{1}_p, \boldsymbol{\Sigma})$ and $\boldsymbol{\Sigma} = \boldsymbol{Z}_3 \boldsymbol{V}_3 \boldsymbol{Z}_3' + \boldsymbol{V}_2 + \sigma^2 \boldsymbol{I}_p$.
- $\boldsymbol{Y} \sim N_{p,n}(\mu \mathbf{1}_p \mathbf{1}_n', \boldsymbol{\Sigma}, \boldsymbol{I}_n)$, where $\boldsymbol{Y} = (\mathbf{y}_1 : \mathbf{y}_2 : \ldots : \mathbf{y}_n)$ are $n$ independent samples.

# External test

Hypotheses at "macro-level": test the global structures of $\boldsymbol{\Sigma}$

- $H_1 : \boldsymbol{\Sigma}_I = \boldsymbol{I}_{n_2} \otimes \boldsymbol{\Sigma}_1 + (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes \boldsymbol{\Sigma}_2$, where $\boldsymbol{\Sigma}_h$, $h = 1, 2$, is a $n_1 \times n_1$ unstructured matrix.
- $H_2 : \boldsymbol{\Sigma}_{II} = \boldsymbol{I}_{n_2} \otimes \boldsymbol{\Sigma}_1 + (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes \boldsymbol{\Sigma}_2$, where $\boldsymbol{\Sigma}_h$, $h = 1, 2$, is a CT matrix and depends on $r$ parameters, $r = [n_1/2] + 1$.
- $H_3 : \boldsymbol{\Sigma}_{III} = \boldsymbol{I}_{n_2} \otimes \boldsymbol{\Sigma}_1 + (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes \boldsymbol{\Sigma}_2$, where $\boldsymbol{\Sigma}_h$, $h = 1, 2$, is a CS matrix and can be written as $\boldsymbol{\Sigma}_h = \sigma_{h1} \boldsymbol{I}_{n_1} + \sigma_{h2} (\boldsymbol{J}_{n_1} - \boldsymbol{I}_{n_1})$.

The number of parameters are $n_1(n_1 + 1)$, $2r$ and $4$, respectively.

If $n_2 = 4, n_1 = 4$, then

$$
\left(\begin{array}{cccc|cccc|cccc|cccc}
\tau_0 & \tau_1 & \tau_2 & \tau_1 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 \\
\tau_1 & \tau_0 & \tau_1 & \tau_2 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 \\
\tau_2 & \tau_1 & \tau_0 & \tau_1 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 \\
\tau_1 & \tau_2 & \tau_1 & \tau_0 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 \\
\hline
\tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_0 & \tau_1 & \tau_2 & \tau_1 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 \\
\tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_1 & \tau_0 & \tau_1 & \tau_2 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 \\
\tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_2 & \tau_1 & \tau_0 & \tau_1 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 \\
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\tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_0 & \tau_1 & \tau_2 & \tau_1 & \tau_3 & \tau_4 & \tau_5 & \tau_4 \\
\tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_1 & \tau_0 & \tau_1 & \tau_2 & \tau_4 & \tau_3 & \tau_4 & \tau_5 \\
\tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_2 & \tau_1 & \tau_0 & \tau_1 & \tau_5 & \tau_4 & \tau_3 & \tau_4 \\
\tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_1 & \tau_2 & \tau_1 & \tau_0 & \tau_4 & \tau_5 & \tau_4 & \tau_3 \\
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\tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_1 & \tau_0 & \tau_1 & \tau_2 \\
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\tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_1 & \tau_2 & \tau_1 & \tau_0
\end{array}\right).
$$

# Selected previous work of symmetry model

- Olkin and Press (1969), Olkin (1973)
- Andersson (1975), Perlman (1987)
- Nahtman (2006) and Nahtman and von Rosen (2008) studied properties of some patterned covariance matrices arising under different symmetry restrictions in balanced mixed linear models.
- Roy and Fonseca (2012): double exchangeability

# Canonical reduction and equivalent hypotheses

**Lemma**

*(Arnold, 1973) Suppose* $\boldsymbol{Y} \sim N_{p,n}(\mu \mathbf{1}_p \mathbf{1}'_n, \boldsymbol{\Sigma}_I, \boldsymbol{I}_n)$, *where* $p = n_2 n_1$ *and* $\mu$ *is an unknown scalar parameter. Let* $\boldsymbol{\Gamma}_1$ *be an* $n_2 \times n_2$ *orthogonal matrix whose first column is proportional to* $\mathbf{1}_{n_1}$ *and put* $(\boldsymbol{Y}'_1 \vdots \boldsymbol{Y}'_2)' = (\boldsymbol{\Gamma}'_1 \otimes \boldsymbol{I}_{n_1})\boldsymbol{Y}$, *where* $\boldsymbol{Y}_1$: $n_1 \times n$ *and* $\boldsymbol{Y}_2$: $n_1(n_2-1) \times n$. *Then,* $\boldsymbol{Y}_1$ *and* $\boldsymbol{Y}_2$ *are independently distributed, and*

$$\boldsymbol{Y}_1 \sim N_{n_1,n}\left(\sqrt{n_2}\mu \mathbf{1}_{n_1}\mathbf{1}'_n, \boldsymbol{\Delta}_1, \boldsymbol{I}_n\right), \quad (1)$$

$$\boldsymbol{Y}_2 \sim N_{n_1(n_2-1),n}\left(\mathbf{0}, \boldsymbol{I}_{n_2-1} \otimes \boldsymbol{\Delta}_2, \boldsymbol{I}_n\right), \quad (2)$$

*where* $\boldsymbol{\Delta}_1 = \boldsymbol{\Sigma}_1 + (n_2-1)\boldsymbol{\Sigma}_2$ *and* $\boldsymbol{\Delta}_2 = \boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2$. *Moreover,* $\boldsymbol{\Sigma}_I$ *is positive definite if and only if both* $\boldsymbol{\Delta}_1$ *and* $\boldsymbol{\Delta}_2$ *are positive definite.*

## Theorem

*Let* $\boldsymbol{\Gamma}_2$ *be the orthogonal matrix whose columns* $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n_1}$ *are the known orthonormal eigenvectors of any* $n_1 \times n_1$ *CT matrices. To test* $H_i$, $i = 1, 2, 3$, *is respectively equivalent to test*

$$H_1: \quad (\boldsymbol{\Gamma}_1' \otimes \boldsymbol{I}_{n_1})\boldsymbol{\Sigma}(\boldsymbol{\Gamma}_1 \otimes \boldsymbol{I}_{n_1}) = block - diag(\boldsymbol{\Delta}_1, \boldsymbol{\Delta}_2 = \cdots = \boldsymbol{\Delta}_2),$$

$$H_2: \quad \boldsymbol{\Gamma}_2'\boldsymbol{\Delta}_1\boldsymbol{\Gamma}_2 = diag(\lambda_1, \cdots, \lambda_{n_1}), \lambda_i = \lambda_{n_1-i+2},$$

$$\boldsymbol{\Gamma}_2'\boldsymbol{\Lambda}_2\boldsymbol{\Gamma}_2 = \cdots = \boldsymbol{\Gamma}_2'\boldsymbol{\Delta}_2\boldsymbol{\Gamma}_2$$

$$= diag(\lambda_{n_1+1}, \cdots, \lambda_{2n_1}) = \cdots = diag(\lambda_{(n_2-1)n_1+1}, \cdots, \lambda_{n_2 n_1})$$

*and* $\lambda_{(j-1)n_1+i} = \lambda_{jn_1-i+2}, i = 2, \ldots, [n_1/2] + 1, j = 2, \ldots, n_2,$

*assuming* $H_1$,

$$H_3: \quad \lambda_2 = \cdots = \lambda_{n_1}, \lambda_{n_1+1} = \lambda_{2n_1+1} = \cdots = \lambda_{(n_2-1)n_1+1},$$

$$\lambda_{n_1+2} = \cdots = \lambda_{2n_1} = \cdots = \lambda_{(n_2-1)n_1+2} = \cdots = \lambda_{n_2 n_1}$$

*assuming* $H_2$.

# Likelihood ratio test

$H_0 : \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{II}$ and $H_A : \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_I$,
i.e. test one exchangeable hierarchy in data and the other level is circularly correlated versus one exchangeable hierarchy and the other unstructured level in data.

$$\lambda_{21} = \left( \frac{\left| \frac{\boldsymbol{T}_2}{n(n_2-1)} \right|^{n_2-1} \left| \frac{\boldsymbol{T}_1}{n} \right|}{\prod_{i=1}^{2r} \left( \frac{t_{2i}}{nm_i} \right)^{m_i}} \right)^{n/2},$$

where $t_{21} = tr(\boldsymbol{T}_1 \boldsymbol{P}_{n_1})$, $t_{2i} = tr(\boldsymbol{T}_1(\boldsymbol{v}_i\boldsymbol{v}_i' + \boldsymbol{v}_{n_1-i+2}\boldsymbol{v}_{n_1-i+2}'))$,
$t_{2(r+1)} = tr(\boldsymbol{T}_2 \boldsymbol{P}_{n_1})$ and
$t_{2(r+i)} = tr(\boldsymbol{T}_2(\boldsymbol{v}_i\boldsymbol{v}_i' + \boldsymbol{v}_{n_1-i+2}\boldsymbol{v}_{n_1-i+2}'))$, $i = 2, \dots, r$.

# LRT, cont.

$H_0 : \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{III}$ and $H_A : \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_I$,
i.e. test the doubly exchangeable hierarchical data versus one exchangeable hierarchy and the other unstructured level in data.

$$\lambda_{31} = \left( \frac{\left| \frac{\boldsymbol{T}_2}{n(n_2-1)} \right|^{n_2-1} \left| \frac{\boldsymbol{T}_1}{n} \right|}{\prod_{j=1}^{4} \left( \frac{t_{3j}}{nm_j} \right)^{m_j}} \right)^{n/2},$$

where $t_{31} = tr(\boldsymbol{T}_1\boldsymbol{P}_{n_1})$, $t_{32} = tr(\boldsymbol{T}_1\boldsymbol{Q}_{n_1})$, $t_{33} = tr(\boldsymbol{T}_2\boldsymbol{P}_{n_1})$ and $t_{34} = tr(\boldsymbol{T}_2\boldsymbol{Q}_{n_1})$.

# LRT, cont.

$H_0 : \mathbf{\Sigma} = \mathbf{\Sigma}_{III}$ and $H_A : \mathbf{\Sigma} = \mathbf{\Sigma}_{II}$,
i.e. test the doubly exchangeable hierarchical data versus one exchangeable hierarchy in data and the other level is circularly correlated.

$$\lambda_{32} = \left( \frac{\prod_{i=2}^{r} \left(\frac{t_{2i}}{nm_i}\right)^{m_i} \prod_{i=r+2}^{2r} \left(\frac{t_{2i}}{nm_i}\right)^{m_i}}{\left(\frac{t_{32}}{n(n_1-1)}\right)^{n_1-1} \left(\frac{t_{34}}{n(n_2-1)(n_1-1)}\right)^{(n_2-1)(n_1-1)}} \right)^{n/2} .$$

# $n_2 = 3$, $n_1 = 4$

Tabela : Type I error probabilities of LRT for $\alpha = 5\%$

| $n$ | $\lambda_{21}$ | $\lambda_{32}$ | $\lambda_{31}$ |
|---|---|---|---|
| 5 | 0.614 | 0.075 | 0.613 |
| 10 | 0.191 | 0.067 | 0.197 |
| 20 | 0.093 | 0.053 | 0.109 |
| 30 | 0.078 | 0.056 | 0.075 |
| 40 | 0.071 | 0.050 | 0.077 |
| 50 | 0.063 | 0.054 | 0.060 |
| 60 | 0.064 | 0.047 | 0.057 |
| 70 | 0.064 | 0.055 | 0.065 |
| 80 | 0.059 | 0.046 | 0.056 |
| 90 | 0.058 | 0.040 | 0.051 |
| 100 | 0.045 | 0.057 | 0.054 |

# Internal test

Hypotheses at “micro-level”: testing the specific variance components given $\boldsymbol{\Sigma}$ has a block circular structure.

- Assuming $\boldsymbol{V}_3 = \sigma_1 \boldsymbol{I}_{n_2} + \sigma_2(\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2})$ and $\boldsymbol{V}_2$ has the same structure of $\boldsymbol{\Sigma}_{II}$.
- $\tilde{\boldsymbol{\Sigma}} = \boldsymbol{I}_{n_2} \otimes \tilde{\boldsymbol{\Sigma}}_1 + (\boldsymbol{J}_{n_2} - \boldsymbol{I}_{n_2}) \otimes \tilde{\boldsymbol{\Sigma}}_2$, where $\tilde{\boldsymbol{\Sigma}}_h$, $h = 1, 2$, is also a CT matrix.
- $\tilde{\boldsymbol{\Sigma}}_1 = Toep(\sigma^2 + \sigma_1 + \tau_1, \sigma_1 + \tau_2, \ldots, \sigma_1 + \tau_2)$ and $\tilde{\boldsymbol{\Sigma}}_2 = Toep(\sigma_2 + \tau_{r+1}, \sigma_2 + \tau_{r+2}, \ldots, \sigma_2 + \tau_{r+2})$.
- $\boldsymbol{\theta} = (\sigma^2, \sigma_1, \sigma_2, \tau_1, \ldots, \tau_{2r})'$ and $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_{2r})'$

# Remarks

- The model does not have explicit expression since the number of unknown parameters in $\mathbf{\Sigma}$ $(2r+3)$ is more than the number of distinct eigenvalues of $\mathbf{\Sigma}$ $(2r)$. (Szatrowski, 1980)
- Different restricted models are considered. (Liang et al., 2014)
- Testing hypotheses means to impose restrictions on restricted models.
- possible to derive equivalent hypotheses through the distinct eigenvalues $\boldsymbol{\eta}$

# Equivalent hypotheses

## Theorem

*For the model under the restriction $\boldsymbol{K}_1\boldsymbol{\theta} = \mathbf{0}$, the following hypotheses are equivalent:*
(i) $\sigma_2 = 0$,
(ii) $\eta_1 = \eta_{r+1}$.

$$\lambda_1^{2/n} \overset{d}{\sim} \begin{cases} X(1-X)^{n_2-1}, & \text{with probability } P\left[F(n-1, n(n_2-1)) \leqslant \frac{n}{n-1}\right], \\ 1, & \text{with probability } 1 - P\left[F(n-1, n(n_2-1)) \leqslant \frac{n}{n-1}\right] \end{cases}$$

where $X \sim Beta(\frac{n-1}{2}, \frac{n(n_2-1)}{2})$.

### Theorem

*For the model under the restriction $\boldsymbol{K}_1\boldsymbol{\theta} = \mathbf{0}$, the following hypotheses are equivalent:*
(i) $\tau_2 = \ldots = \tau_r$ and $\tau_{r+2} = \ldots = \tau_{2r}$,
(ii) $\eta_2 = \ldots = \eta_r$ and $\eta_{r+2} = \ldots = \eta_{2r}$.

$$\lambda_2^{2/n} \overset{d}{\sim} \prod_i B_i^{m_i}, \quad i \in \{2, \ldots, r\} \cup \{r+2, \ldots, 2r\}.$$

where $B_i \sim Beta(\frac{nm_i}{2}, \frac{n(\sum_{i=2}^{r} m_i - m_i)}{2})$.

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Thank you for your attention!