Hypothesis testing in variance components with constraints

M. Przystalski[1] M. Fonseca[2]

[1]The Research Center for Cultivar Testing, Słupia Wielka, Poland

[2]Centro de Mathemática e Aplicações, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Lisboba, Portugal

LinStat 2014

Introduction

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- E.g. Self and Liang (1987) had shown that this not necessarily hold.
- For LMM with one variance component, Crainiceanu and Ruppert (2004) had shown that the distribution of the LR statistic is mixture of two chi-squares.
- This is due to the fact that the values of the variance component under the null lie on the boundry of the parameter space.

Introduction

Aims

1. Construct statiscal tests for variance components with posistivity constraints.

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2. Investigate the perfomance of the proposed procedures.

$$\boldsymbol{y} = \mu\mathbf{1} + \mathbf{Z}\boldsymbol{\alpha} + \boldsymbol{e},$$

where

- $\boldsymbol{y}$ - vector of observed values,
- μ - general mean
- $\mathbf{1}_N$ - vector of one's
- $\boldsymbol{\alpha}$ - vector of random effects,
- $\mathbf{Z} = \mathbf{I}_a \otimes \mathbf{1}_n$ is design matrix,
- $\boldsymbol{e}$ - vector of errors

We assume that $\boldsymbol{\alpha}$ and $\boldsymbol{e}$ are indendent and that $\boldsymbol{\alpha} \sim N\left(\mathbf{0}, \sigma_a^2\mathbf{I}_a\right)$ and $\boldsymbol{e} \sim N\left(\mathbf{0}, \sigma_e^2\mathbf{I}_N\right)$,respectively.

- The problem of interest is:

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where $\gamma_1 = n\sigma_a^2 + \sigma_e^2$ and $\gamma_2 = \sigma_e^2$ (see Khuri et al., 1998).

- The maximum likelihood estimators (MLE)

$$\widehat{\mu} = \frac{1}{N}\mathbf{1}_N'\boldsymbol{y}, \qquad \widehat{\gamma}_1 = \frac{1}{a-1}\boldsymbol{y}'\mathbf{P}_1\boldsymbol{y} \qquad \widehat{\gamma}_2 = \frac{1}{N-a}\boldsymbol{y}'\mathbf{P}_2\boldsymbol{y},$$

where

$$\mathbf{P}_1 = \frac{\mathbf{I}_a \otimes \mathbf{J}_n}{n} - \frac{\mathbf{J}_N}{N} \qquad \mathbf{P}_2 = \mathbf{I}_a \otimes \mathbf{I}_n - \frac{\mathbf{I}_a \otimes \mathbf{J}_n}{n}.$$

One-way model
Statistical model

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- $$\mathbf{V}_{\widehat{\boldsymbol{\gamma}}}^{-1/2}\left(\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}\right) \xrightarrow{d} N\left(\mathbf{0}, \mathbf{I}_2\right). \qquad (1)$$

One-way model
Likelihood ratio

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$$\frac{d\ell_{12}}{d\gamma^{-1}} = -\widehat{\gamma}_1 (a-1)\left(\frac{\widehat{\gamma}_1}{\gamma} - 1\right) - \widehat{\gamma}_2 (N-a)\left(\frac{\widehat{\gamma}_2}{\gamma} - 1\right) = 0.$$

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One-way model
Likelihood ratio

Proposition

The LRT rejects the null hypothesis for large values of λ given by:

$$\lambda = \begin{cases} 0 & \widehat{\gamma}_1 \leq \widehat{\gamma}_2 \\ \frac{(a-1)(N-a)(\widehat{\gamma}_1 - \widehat{\gamma}_2)^2}{\left((a-1)\widehat{\gamma}_1^2 + (N-a)\widehat{\gamma}_2^2\right)} & \widehat{\gamma}_1 > \widehat{\gamma}_2. \end{cases}$$

$$\boldsymbol{y} = [y_{ij}] \text{ -- lexicographical order}$$

- i – first factor
- j – replicates

Permutation

$$\boldsymbol{y}_l = \mathbf{P}_l \boldsymbol{y},$$

where $\mathbf{P}_l$ is a random permutation matrix.

1. Calculate $\hat{\gamma}_1$ and $\hat{\gamma}_2$
2. Calculate λ
3. Generate a pseudo-sample of size M of

$$\ell_l = \ell(\boldsymbol{y}_l) = \ell(\mathbf{P}_l \boldsymbol{y})$$

4. Retrieve c_α, the pseudo-sample's empirical $(1-\alpha)$-th quantile
5. Compare with the observed value λ with c_α

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One-way model
Application of delta method

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Theorem

The distribution of the LRT κ is a mixture of χ^2_0 and χ^2_1 with coefficients p and 1-p, with

$$p = P\left(\widehat{\gamma}_1 \leq \widehat{\gamma}_2\right).$$

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Statistical model

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where $\gamma_1 = br\sigma_a^2 + r\sigma_b^2 + \sigma_e^2$, $\gamma_2 = r\sigma_b^2 + \sigma_e^2$ and $\gamma_3 = \sigma_e^2$, respectively (see Khuri et al, 1998).

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Two-fold nested model
Statistical model

- The maximum likelihood estimators (MLE)

$$\widehat{\mu} = \frac{1}{N}\mathbf{1}_N'\boldsymbol{y},\ \widehat{\gamma}_1 = \frac{\boldsymbol{y}'\mathbf{P}_1\boldsymbol{y}}{a-1},\ \widehat{\gamma}_2 = \frac{\boldsymbol{y}\mathbf{P}_2\boldsymbol{y}}{a\,(b-1)},\ \widehat{\gamma}_3 = \frac{\boldsymbol{y}'\mathbf{P}_3\boldsymbol{y}}{ab\,(r-1)},$$

  where $\mathbf{P}_1 = \left(\mathbf{I}_a - \frac{1}{a}\mathbf{J}_a\right) \otimes \frac{1}{br}\mathbf{J}_{br}$, $\mathbf{P}_2 = \mathbf{I}_a \otimes \left(\mathbf{I}_b - \frac{1}{b}\mathbf{J}_b\right) \otimes \frac{1}{r}\mathbf{J}_r$, $\mathbf{P}_3 = \mathbf{I}_{ab} \otimes \left(\mathbf{I}_r - \frac{1}{r}\mathbf{J}_r\right)$.
- $E\left(\widehat{\gamma}_1\right) = \gamma_1$ and $V\left(\widehat{\gamma}_1\right) \approx \frac{2\gamma_1^2}{a}$; $E\left(\widehat{\gamma}_2\right) = \gamma_2$ and; $V\left(\widehat{\gamma}_2\right) \approx \frac{2\gamma_2^2}{ab}$; $E\left(\widehat{\gamma}_3\right) = \gamma_3$ and; $V\left(\widehat{\gamma}_3\right) \approx \frac{2\gamma_3^2}{abr}$.
- $\mathbf{V}_{\widehat{\gamma}} = diag\left(\frac{2\gamma_1^2}{a}, \frac{2\gamma_2^2}{ab}, \frac{2\gamma_3^2}{abr}\right)$

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where $\mathbf{P}_1 = \left(\mathbf{I}_a - \frac{1}{a}\mathbf{J}_a\right) \otimes \frac{1}{br}\mathbf{J}_{br}$, $\mathbf{P}_2 = \mathbf{I}_a \otimes \left(\mathbf{I}_b - \frac{1}{b}\mathbf{J}_b\right) \otimes \frac{1}{r}\mathbf{J}_r$, $\mathbf{P}_3 = \mathbf{I}_{ab} \otimes \left(\mathbf{I}_r - \frac{1}{r}\mathbf{J}_r\right)$.

- $E\left(\widehat{\gamma}_1\right) = \gamma_1$ and $V\left(\widehat{\gamma}_1\right) \approx \frac{2\gamma_1^2}{a}$; $E\left(\widehat{\gamma}_2\right) = \gamma_2$ and; $V\left(\widehat{\gamma}_2\right) \approx \frac{2\gamma_2^2}{ab}$; $E\left(\widehat{\gamma}_3\right) = \gamma_3$ and; $V\left(\widehat{\gamma}_3\right) \approx \frac{2\gamma_3^2}{abr}$.
- $\mathbf{V}_{\widehat{\gamma}} = diag\left(\frac{2\gamma_1^2}{a}, \frac{2\gamma_2^2}{ab}, \frac{2\gamma_3^2}{abr}\right)$
-

$$\mathbf{V}_{\widehat{\gamma}}^{-1/2}\left(\widehat{\gamma} - \gamma\right) \xrightarrow{d} N\left(\mathbf{0}, \mathbf{I}_3\right). \tag{2}$$

Two-fold nested model
Likelihood ratio. H_1

$\widehat{\gamma}_1 \geq \widehat{\gamma}_2 \geq \widehat{\gamma}_3$

$$\widetilde{\gamma}_1 = \widehat{\gamma}_1$$
$$\widetilde{\gamma}_2 = \widehat{\gamma}_2$$
$$\widetilde{\gamma}_3 = \widehat{\gamma}_3$$

$$-2\ell_1 = 0$$

Likelihood Ratio

H_1

$\widehat{\gamma}_1 < \widehat{\gamma}_2, \widehat{\gamma}_2 \geq \widehat{\gamma}_3$

$$\widetilde{\gamma}_{12} = \frac{\widehat{\gamma}_1^2 + b\widehat{\gamma}_2^2}{\widehat{\gamma}_1 + b\widehat{\gamma}_2}$$

$$\widetilde{\gamma}_3 = \widehat{\gamma}_3$$

$$-2\ell_1 = \frac{ab^2(\widehat{\gamma}_1\widehat{\gamma}_2 - \widehat{\gamma}_2^2)^2 + ab(\widehat{\gamma}_1\widehat{\gamma}_2 - \widehat{\gamma}_1^2)^2}{(\widehat{\gamma}_1^2 + b\widehat{\gamma}_2^2)^2}$$

Two-fold nested model
Likelihood ratio. H_1

$\widehat{\gamma}_1 \geq \frac{\widehat{\gamma}_2^2 + r\widehat{\gamma}_3^2}{\widehat{\gamma}_2 + r\widehat{\gamma}_3}, \widehat{\gamma}_2 < \widehat{\gamma}_3$

$$\widetilde{\gamma}_1 = \widehat{\gamma}_1$$

$$\widetilde{\gamma}_{23} = \frac{\widehat{\gamma}_2^2 + r\widehat{\gamma}_3^2}{\widehat{\gamma}_2 + r\widehat{\gamma}_3}$$

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$$\widetilde{\gamma}_{123} = \frac{\widehat{\gamma}_1^2 + b\widehat{\gamma}_2^2 + br\widehat{\gamma}_3^2}{\widehat{\gamma}_1 + b\widehat{\gamma}_2 + br\widehat{\gamma}_3}$$

$$\begin{aligned} -2\ell_1 = & \frac{a\Big(b(\widehat{\gamma}_1\widehat{\gamma}_2 - \widehat{\gamma}_2^2) + br(\widehat{\gamma}_1\widehat{\gamma}_3 - \widehat{\gamma}_3^2)\Big)^2}{(\widehat{\gamma}_1^2 + b\widehat{\gamma}_2^2 + br\widehat{\gamma}_3^2)^2} \\ & + \frac{ab\Big((\widehat{\gamma}_1\widehat{\gamma}_2 - \widehat{\gamma}_1^2) + br(\widehat{\gamma}_2\widehat{\gamma}_3 - \widehat{\gamma}_3^2)\Big)^2}{(\widehat{\gamma}_1^2 + b\widehat{\gamma}_2^2 + br\widehat{\gamma}_3^2)^2} \\ & + \frac{abr\Big((\widehat{\gamma}_1\widehat{\gamma}_2 - \widehat{\gamma}_1^2) + b(\widehat{\gamma}_1\widehat{\gamma}_2 - \widehat{\gamma}_2^2)\Big)^2}{(\widehat{\gamma}_1^2 + b\widehat{\gamma}_2^2 + br\widehat{\gamma}_3^2)^2} \end{aligned}$$

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$$\boldsymbol{y} = [y_{ijk}] \text{ – lexicographical order}$$

- i – first factor
- j – second factor
- k – replicates

Selective permutation

$$\boldsymbol{y}_l = (\mathbf{I}_a \otimes \mathbf{P}_l)\boldsymbol{y},$$

where $\mathbf{P}_l$ is a random permutation matrix.

1. Calculate $\hat{\gamma}_1$, $\hat{\gamma}_2$ and $\hat{\gamma}_3$
2. Calculate λ
3. Generate a pseudo-sample of size M of

$$\ell_l = \ell(\mathbf{y}_l) = \ell\big((\mathbf{I}_a \otimes \mathbf{P}_l)\mathbf{y}\big)$$

4. Retrieve c_α, the pseudo-sample's empirical $(1-\alpha)$-th quantile
5. Compare the observed value of λ with c_α

- (i) $N(0, 1)$,

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- $a = 20$, $b = 10$

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- $a = 20$, $b = 10$
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- 1000 simulations runs
- 1000 permutations
- as a reference we took F-test

σ_a^2	$N(0,1),\ N(0,1)$		$G(10,5),\ G(10,5)$		$t-St.(5),\ t-St.(5)$	
	F test	Perm. test	F test	Perm. test	F test	Perm. test
0	0.047	0.049	0.057	0.053	0.047	0.047
0.05	0.121	0.116	0.142	0.140	0.120	0.118
0.1	0.216	0.209	0.249	0.251	0.227	0.228
0.2	0.425	0.426	0.452	0.456	0.413	0.407
0.5	0.811	0.801	0.792	0.793	0.728	0.728
0.75	0.914	0.916	0.905	0.900	0.854	0.851
1	0.951	0.953	0.956	0.953	0.922	0.919

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Similation studies
Two-fold nested model

- $(i)\ N(0,1)$, $(ii)\ G(10,5)$, $(iii)\ t - Stud.(5)$
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- $a = 20,\ b = 5,\ r = 5$
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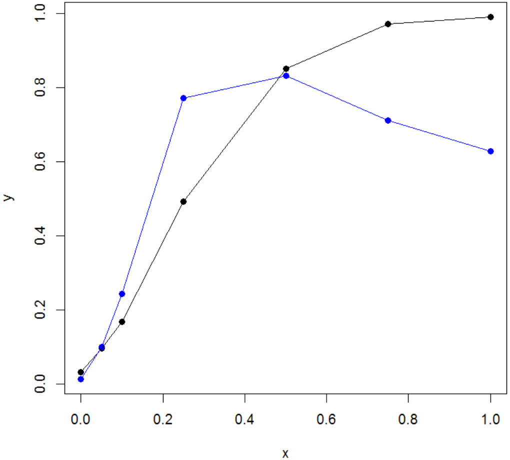
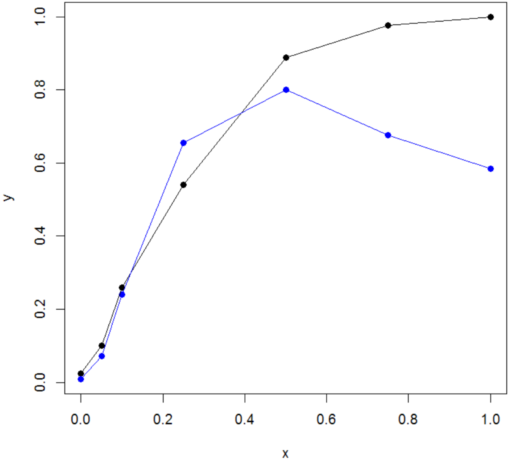
(*i*) $N(0,1)$

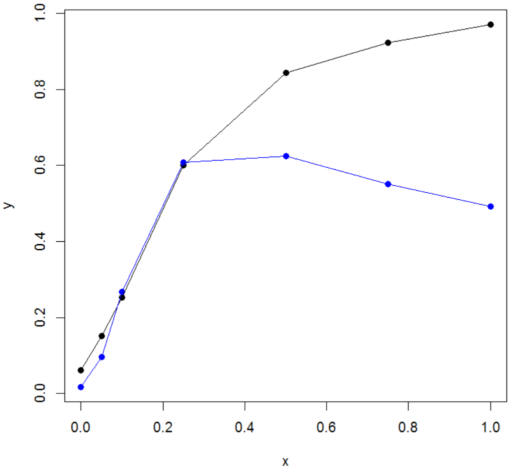


$(ii)\ G(10,1)$



Simulation studies
Two-fold nested model

$(iii)\ t - Stud.(5)$



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  - Perform simulations for $t - Stud$ with increased degrees of freedom.

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