Model of Fatigue Failure Due to Multiple Cracks using Extended Birnbaum-Saunders Distribution

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Birnbaum-Saunders distribution:

Probability distributions that are common in data analysis and in design against fatigue-life include the lognormal distribution, inverse Gaussian distribution, Weibull distribution and the **BS distribution**. In 1969 Birnbaum and Saunders derived **an univariate two-parameter family of life distributions** for the failure time of a material due to fatigue based on an idealized model of the number of cycles necessary to force a fatigue crack to grow past a critical value.

The BS distribution which is commonly known as the **fatigue life distribution** is based on certain assumptions to model the mode in which the failure occurs by the initiation, expansion and then eventual extension of a dominant crack past some critical threshold ω for the first time.

This presentation introduces an extension of the Birnbaum–Saunders (BS) distribution which is motivated by the thermo-mechanical fatigue (TMF) failure of materials with complex heterogeneous microstructures such as cast iron.

Why the extension of the Birnbaum-Saunders distribution?

Many mechanical structural components in most fields of engineering are subject to fatigue at elevated temperatures. Components operating under conditions which involve both thermal and mechanical loads are frequently subject to TMF. TMF is the fatigue failure of a material that arises due to thermal related stresses that take place during its normal operating conditions.

A typical example of TMF failure can be found in the cylinder head of truck engines which, due to their operating conditions, are subject to frequent temperature changes. As soon as the engine starts up, the temperature increases until it reaches the full operating temperature. This increase in temperature makes the material to expand. However, due to the constrained condition found in the cylinder head, compressive stresses developed, that can lead to plastic deformation of the material.

When the engine is shut down, the temperature decreases and the compressive stresses are relieved. Nonetheless, due to the plastic deformation in compression, residual tensile stresses develop. During the engine life this cycle is continuously repeated. **The repetition of the "start up - shut down" cycle in the end could lead to localized cracking of the truck engine cylinder head.**

The extension of the Birnbaum-Saunders distribution?

Taking into account that the inclusion (graphite) volume fraction is known and specified for a given material, one may assume the number of cracks is a known and fixed number m.

The m microcracks form multiple larger cracks that ultimately lead to failure of the material.

Our new EBS distribution for m number of cracks has parameters $\boldsymbol{A}$ and $\boldsymbol{\beta}$ as the shape matrix and the scale vector respectively, as compared to the scalar quantities α and β as the shape and the scale parameters of the traditional BS distribution.

We show through many remarks throughout the article that for $m = 1$ our new EBS distribution reduces to the conventional BS distribution.

We also show with the help of simulation study that our EBS is a better model than the BS model in the sense of parameter estimation (more precision and less standard deviation) when we have multiple crack information in the model.

We develop an extension of the Birnbaum–Saunders distribution under some basic assumptions to model the fatigue failure time of a material due to the growth of these m cracks.

Genesis of the Birnbaum-Saunders model for multiple cracks:

- **We assume that the crack propagation rate is the same for all m cracks.**
- **The cracks originated in the first cycle have the highest chance to lead to failure of the material.**
- **Thus, we assume that the m cracks are initiated from the very first cycle.**
- **Let $y_{i,j}$ represents the total crack extension for the j^{th} crack $(j = 1, \ldots, m)$ in the i^{th} cycle.**
- **Under similar assumption of the Birnbaum-Saunders (1969a) article, but allowing dependence among the m random cracks in the same cycle, we assume that the vector $\boldsymbol{y}_i = (y_{i,1}, \ldots, y_{i,m})' \sim N_m(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_m)'$, with $\mu_j > 0$, for $j = 1, \ldots, m$, and $\boldsymbol{\Sigma}$, a positive definite matrix, which does not depend on the cycle under consideration, i.e., it is same for all the cycles.**
- **Moreover, vectors $\boldsymbol{y}_i : i \in \Im$ are independent.**
- **For each cycle k we assume that the random vector $\boldsymbol{s}_k = (s_{k,1}, \ldots, s_{k,m})'$, where $s_{k,j} = \sum_{i=1}^{k} y_{i,j}$ denotes the size of the j^{th} crack after the k^{th} cycle.**
- **It is clear that under these assumptions $\boldsymbol{s}_k = \sum_{i=1}^{k} \boldsymbol{y}_i \sim N_m(k\boldsymbol{\mu}, k\boldsymbol{\Sigma})$.**

Genesis of the Birnbaum-Saunders model for multiple cracks:

- **For the smooth functioning of any system the size of the j^{th} crack must not exceed a fixed threshold, the threshold whose particular value $\omega_j > 0$ depends on where this crack is located in the material.**
- **The system collapses if the size of any of these m cracks is greater than its threshold for the first time.**
- **Therefore, the interest is in getting the distribution of the random variable $T = \min\{k : s_{k,1} > \omega_1 \vee \ldots \vee s_{k,m} > \omega_m\}$, where the symbol $\vee$ denotes the logical disjunction "or", that is, the probability $P[T \leq n]$.**
- **Noting that the event $T > n$ occurs if and only if the event $\{s_{n,1} \leq \omega_1 \wedge \ldots \wedge s_{n,m} \leq \omega_m\}$ occurs, where the symbol $\wedge$ denotes the logical conjunction "and", it is clear that**

$$\begin{aligned} P[T \leq n] &= 1 - P[s_{n,1} \leq \omega_1 \wedge \ldots \wedge s_{n,m} \leq \omega_m] \\ &= 1 - \Phi_m\left(n^{-\frac{1}{2}} \boldsymbol{\Sigma}^{-\frac{1}{2}} (\boldsymbol{\omega} - n\boldsymbol{\mu})\right), \end{aligned}$$

where $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)'$, and Φ_m is the m-variate standard multinormal distribution $N_m(\mathbf{0}, \boldsymbol{I}_m)$, with $\boldsymbol{I}_m$ is the m-dimensional identity matrix.

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  where $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)'$, and Φ_m is the m-variate standard multinormal distribution $N_m\left(\mathbf{0}, \boldsymbol{I}_m\right)$, with $\boldsymbol{I}_m$ is the m-dimensional identity matrix.

- It turns out that

$$n^{-\frac{1}{2}}\boldsymbol{\Sigma}^{-\frac{1}{2}}\left(\boldsymbol{\omega} - \boldsymbol{\mu}\right) = \boldsymbol{A} \cdot \left[\left(\sqrt{\frac{\beta_j}{n}} - \sqrt{\frac{n}{\beta_j}}\right)_{j=1}^{m}\right]',$$

  where $\boldsymbol{A} = \left(a_{jh}\right)_{j,h=1}^{m} = \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathrm{diag}\left(\boldsymbol{\omega}^{\frac{1}{2}}\right)\mathrm{diag}\left(\boldsymbol{\mu}^{\frac{1}{2}}\right)$ is a positive definite matrix, and $\beta_j = \frac{\omega_j}{\mu_j}$, for $j = 1, \ldots, m$.

Definition of Extended Birnbaum-Saunders (EBS) Distribution:

Definition 1. *An* $m \times m$ *square matrix* $\boldsymbol{A}$ *has the "positive persistence" condition (property) if and only if* $\forall\ \boldsymbol{x} \in \Re^m : \boldsymbol{x} > \mathbf{0} \Longrightarrow \boldsymbol{A} \cdot \boldsymbol{x} > \mathbf{0}$.

Definition 2. *Given the m-variate vector* $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m)$ *with positive real components* $\beta_j \in \Re_+$ *for* $j = 1, \ldots, m$, *and given the positive definite matrix* $\boldsymbol{A} = (a_{jh})_{j,h=1}^{m}$ *having (satisfying) the positive persistence property, the non negative random variable* T *has the extended Birnbaum-Saunders (EBS) distribution with parameters* $\boldsymbol{A}$ *(shape matrix) and* $\boldsymbol{\beta}$ *(scale vector), denoted by* $T \sim EBS\,(\boldsymbol{A}, \boldsymbol{\beta})$, *if its distribution function* F_T *is given by*

$$F_T\,(t) = P\,[T \leq t] = 1 - \Phi_m\,(\boldsymbol{A} \cdot \boldsymbol{r}\,(t))\, I_{\Re_+}\,(t)\,,$$

where $I_{\Re_+}$ *is the indicator function of the set of the positive real numbers, that is*

$$I_{\Re_+}\,(t) = \begin{cases} 1 & if \quad t \in \Re_+ \\ 0 & if \quad t \notin \Re_+ \end{cases},$$

where $\boldsymbol{r} : \Re_+ \longrightarrow \Re^m$, *the real vectorial function is given by*

$$\boldsymbol{r}\,(t) = (r_1\,(t)\,, \ldots, r_m\,(t))' = \left(\ \sqrt{\frac{\beta_1}{t}} - \sqrt{\frac{t}{\beta_1}}\ \ , \ldots ,\ \ \sqrt{\frac{\beta_m}{t}} - \sqrt{\frac{t}{\beta_m}}\ \right)'.$$

Density function of the Extended Birnbaum-Saunders (EBS) Distribution:

The "vector valued real function" $\boldsymbol{u} : \Re_{+} \longrightarrow \Re^{m}$ defined by

$$\begin{aligned}\boldsymbol{u}(t) &= (u_1(t), \ldots, u_m(t))' = \boldsymbol{A} \boldsymbol{\cdot} \boldsymbol{r}(t) \\ &= \boldsymbol{A \cdot} (r_1(t), \ldots, r_m(t))' = \boldsymbol{A \cdot} \left[\left(\sqrt{\frac{\beta_j}{t}} - \sqrt{\frac{t}{\beta_j}} \right)_{j=1}^{m} \right]',\end{aligned}$$

Now, the density function f_T corresponding to the distribution function $F_T(t)$ is given by

$$\begin{aligned}f_T(t) &= \frac{d}{dt} F_T(t) = -\left[\nabla \Phi_m(\boldsymbol{u}(t))\right]' \cdot \frac{d}{dt} \boldsymbol{u}(t)\, I_{\Re_+}(t) \\ &= \Phi_m(\boldsymbol{u}) \left(\frac{\varphi_1(u_1)}{\Phi_1(u_1)}, \ldots, \frac{\varphi_1(u_m)}{\Phi_1(u_m)} \right) \cdot \boldsymbol{A \cdot} \left[\left(\frac{1}{2\beta_j} \left[\left(\frac{\beta_j}{t} \right)^{\frac{1}{2}} + \left(\frac{\beta_j}{t} \right)^{\frac{3}{2}} \right] \right)_{j=1}^{m} \right]' I_{\Re_+}(t),\end{aligned}$$

where the gradient vector $\nabla \Phi_m(\boldsymbol{u})$ of Φ_m applied to $\boldsymbol{u} = \boldsymbol{u}(t)$ is given by $\nabla \Phi_m(\boldsymbol{u}) = \left(\frac{\partial}{\partial u_1} \Phi_m(\boldsymbol{u}), \ldots, \frac{\partial}{\partial u_m} \Phi_m(\boldsymbol{u}) \right)'$.

Remark:

For $m=1$, the above density function $f_T(t)$ of the EBS random variable T reduces to

$$\begin{aligned} f_T(t) &= \Phi_1(u_1)\left(\frac{\varphi_1(u_1)}{\Phi_1(u_1)}\right)\cdot\frac{1}{\alpha}\cdot\frac{1}{2\beta_1}\left[\left(\frac{\beta_1}{t}\right)^{\frac{1}{2}}+\left(\frac{\beta_1}{t}\right)^{\frac{3}{2}}\right] I_{\Re_+}(t) \\ &= \varphi_1(u_1)\cdot\frac{1}{\alpha}\cdot\frac{1}{2\beta_1}\left[\left(\frac{\beta_1}{t}\right)^{\frac{1}{2}}+\left(\frac{\beta_1}{t}\right)^{\frac{3}{2}}\right] I_{\Re_+}(t), \end{aligned}$$

where $\boldsymbol{A}=\frac{1}{\alpha}$ and $u_1=u_1(t)=\frac{1}{\alpha}\left(\sqrt{\frac{\beta_1}{t}}-\sqrt{\frac{t}{\beta_1}}\right)$ for $m=1$.

Therefore, the above density function becomes

$$f_T(t)=\frac{1}{2\sqrt{2\pi}\alpha\beta_1}\left[\left(\frac{\beta_1}{t}\right)^{\frac{1}{2}}+\left(\frac{\beta_1}{t}\right)^{\frac{3}{2}}\right]\exp\left[-\frac{1}{2\alpha^2}\left(\frac{\beta_1}{t}+\frac{t}{\beta_1}-2\right)\right] I_{\Re_+}(t), \qquad (1)$$

which is the density function of the BS random variable with parameters α and β_1.

Hazard function of the EBS distribution:

The hazard function $h : \Re_+ \longrightarrow \Re_+$ corresponding to the random variable $T \sim EBS\left(\boldsymbol{A}, \boldsymbol{\beta}\right)$ is given by

$$
\begin{aligned}
h\left(t\right) &= \frac{f_T\left(t\right)}{1 - F_T\left(t\right)} = \frac{-\left[\nabla \Phi_m\left(\boldsymbol{u}\left(t\right)\right)\right]' \cdot \frac{d}{dt}\boldsymbol{u}\left(t\right) I_{\Re_+}\left(t\right)}{\Phi_m\left(\boldsymbol{u}\left(t\right)\right)} \\
&= \left[\left(\frac{\varphi_1\left(u_j\right)}{\Phi_1\left(u_j\right)}\right)_{j=1}^{m}\right] \cdot \left[\sum_{i=1}^{m} \frac{a_{ji}}{2t}\left\{\left(\frac{\beta_i}{t}\right)^{\frac{1}{2}} + \left(\frac{t}{\beta_i}\right)^{\frac{1}{2}}\right\}\right]' I_{\Re_+}\left(t\right) \\
&= \sum_{j=1}^{m} h_j\left(t; \boldsymbol{\beta}, \boldsymbol{a}_{j\cdot}\right),
\end{aligned}
\tag{2}
$$

where

$$
h_j\left(t; \boldsymbol{\beta}, \boldsymbol{a}_{j\cdot}\right) = \frac{\varphi_1\left(u_j\right)}{\Phi_1\left(u_j\right)} \sum_{i=1}^{m} \frac{a_{ji}}{2t}\left[\left(\frac{\beta_i}{t}\right)^{\frac{1}{2}} + \left(\frac{t}{\beta_i}\right)^{\frac{1}{2}}\right]. \tag{3}
$$

We thus see from (2) that the hazard function $h\left(t\right)$ of the EBS distribution can be written as a sum of m component functions $h_j\left(t; \boldsymbol{\beta}, \boldsymbol{a}_{j\cdot}\right)$, $j = 1, \ldots, m$. It can be shown that each of these component functions $h_j\left(t; \boldsymbol{\beta}, \boldsymbol{a}_{j\cdot}\right)$ is the hazard function of some distribution function $F_j\left(t; \boldsymbol{\beta}, \boldsymbol{a}_{j\cdot}\right)$ for $j = 1, \ldots, m$.

For this we need to define m distribution functions $F_j\left(t; \boldsymbol{\beta}, \boldsymbol{a}_{j\cdot}\right)$, $j = 1, \ldots, m$, and then derive the corresponding density functions and the hazard functions.

Proposition:

The hazard function $h_j : \Re_+ \longrightarrow \Re$ for $j = 1, \ldots, m$ given by

$$\begin{aligned} h_j\left(t; \boldsymbol{\beta}, \boldsymbol{a}_{j\cdot}\right) &= \frac{d}{dt}\left[-u_j\left(t; \boldsymbol{\beta}, \boldsymbol{a}_{j\cdot}\right)\right] \frac{\varphi_1\left(u_j\left(t; \boldsymbol{\beta}, \boldsymbol{a}_{j\cdot}\right)\right)}{\Phi_1\left(u_j\left(t; \boldsymbol{\beta}, \boldsymbol{a}_{j\cdot}\right)\right)} \\ &= \frac{\varphi_1\left(\sum_{k=1}^m a_{jk} r_k\left(t, \beta_k\right)\right)}{\Phi_1\left(\sum_{k=1}^m a_{jk} r_k\left(t, \beta_k\right)\right)} \sum_{i=1}^m \frac{a_{ji}}{2t}\left[\left(\frac{\beta_i}{t}\right)^{\frac{1}{2}} + \left(\frac{t}{\beta_i}\right)^{\frac{1}{2}}\right], \end{aligned}$$

is an upside down function.

Remark:

For $m = 1$, from (2) we see that the hazard function $h(t)$ of the EBS random variable T reduces to

$$h(t) = h_1\left(t; \beta_1, a_{11}\right) = \frac{d}{dt}\left[-u_1\left(t; \beta_1, a_{11}\right)\right] \frac{\varphi_1\left(u_1\left(t; \beta_1, a_{11}\right)\right)}{\Phi_1\left(u_1\left(t; \beta_1, a_{11}\right)\right)},$$

which is the hazard function of the Birnbaum-Saunders distribution.

Two special cases of the new EBS distribution:

Case 1: $\boldsymbol{A} = \kappa \boldsymbol{I}_m$, $\boldsymbol{\omega} = \omega \mathbf{1}_m$ and $\boldsymbol{\beta} = \beta \mathbf{1}_m$

In this case the density function f_T for $t \in \Re_+$ is given by

$$\begin{aligned} f_T(t) &= \frac{d}{dt} F_T(t) = -m\kappa \varphi_1(\kappa r(t)) \left[\Phi_1(\kappa r(t))\right]^{m-1} \frac{dr(t)}{dt} \\ &= \frac{m\kappa}{2} \varphi_1(\kappa r(t)) \left[\Phi_1(\kappa r(t))\right]^{m-1} \left\{ \frac{1}{\beta} \left[\left(\frac{\beta}{t}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{3}{2}} \right] \right\}. \end{aligned}$$

This above density function f_T is related to the Balakrishnan skew normal (BSN) distribution (proposed by Balakrishnan as a discussant of Arnold and Beaver, 2002; see also Gupta and Gupta, 2004), denoted by $Y \sim BSN(\lambda)$, which is a generalization of the skew normal distribution proposed by Azzalini (1985). A random variable $Y \sim BSN(\lambda)$ has a BSN distribution if its probability density function is given by

$$f_Y(y; \lambda) = \varphi_1(y) \left[\Phi_1(\lambda y)\right]^n \frac{1}{C_n(\lambda)},$$

where n is a positive integer and the skewness parameter $\lambda \in \Re$ and $C_n(\lambda)$ is given by

$$C_n(\lambda) = \int_{-\infty}^{+\infty} \varphi_1(y) \left[\Phi_1(\lambda y)\right]^n dy.$$

The ML estimators $\widehat{\kappa}$ and $\widehat{\beta}$ of κ and β:

The ML estimators $\widehat{\kappa}$ and $\widehat{\beta}$ of κ and β are obtained by simultaneously and iteratively solving the following two equations.

$$0 = \frac{n}{\kappa} - \kappa \sum_{i=1}^{n} \left[\frac{\beta}{t_i} + \frac{t_i}{\beta} - 2 \right] + (m-1) \sum_{i=1}^{n} \frac{\varphi_1 \left(\kappa r \left(t_i, \beta \right) \right)}{\Phi_1 \left(\kappa r \left(t_i, \beta \right) \right)} \left(\frac{\beta^{\frac{1}{2}}}{t^{\frac{1}{2}}} - \frac{t^{\frac{1}{2}}}{\beta^{\frac{1}{2}}} \right), \tag{4}$$

and

$$0 = \frac{\kappa}{2\beta} \left\{ -\beta\kappa \sum_{i=1}^{n} \frac{1}{t_i} + \frac{\kappa}{\beta} \sum_{i=1}^{n} t_i + (m-1) \sum_{i=1}^{n} \frac{\varphi_1 \left(\kappa r \left(t_i, \beta \right) \right)}{\Phi_1 \left(\kappa r \left(t_i, \beta \right) \right)} \left(\frac{\beta^{\frac{1}{2}}}{t^{\frac{1}{2}}} + \frac{t^{\frac{1}{2}}}{\beta^{\frac{1}{2}}} \right) - \frac{n}{\kappa} + \frac{2\beta}{k} \sum_{i=1}^{n} \frac{1}{t_i + \beta} \right\}. \tag{5}$$

Remark:

We have observed when $m = 1$ the equation (4) reduces to

$$\frac{1}{\kappa^2} = \frac{\beta}{g} + \frac{s}{\beta} - 2, \tag{6}$$

where

$$g = \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{t_i} \right)^{-1}, \tag{7}$$

and

$$s = \frac{1}{n} \sum_{i=1}^{n} t_i. \tag{8}$$

Remark Continued:

Birnbaum and Saunders (1969b) also showed that the MLE of α in terms of MLE of β can be written as equation (6), where $\alpha = \frac{1}{\kappa}$. Now, for $m = 1$ the equation (5) reduces to

$$\frac{1}{g} - \frac{s}{\beta^2} + \frac{1}{\beta\kappa^2} - \frac{2}{\kappa^2 K(\beta)} = 0, \tag{9}$$

where

$$K(\beta) = \left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{t_i + \beta}\right]^{-1}. \tag{10}$$

Substituting the value of $\frac{1}{\kappa^2}$ from (6) in the equation (9) and simplifying we get

$$\beta^2 - [K(\beta) + 2g]\,\beta + [K(\beta) + s]\,g = 0. \tag{11}$$

That is, for $m = 1$ the the MLE of β can be obtained as the positive root of the above equation (11). Once the MLE of β is obtained, the MLE of κ can be obtained as an exact solution of the equation (6). Birnbaum and Saunders (1969b) also obtained the same nonlinear equation (11), which they tried to solve by two iterative methods, but they noticed that their methods did not work well for all values of α. The same equations (6) and (11) were also obtained by Lemonte, Cribari-Neto and Vasconcellos (2007), in which they proposed to find the MLEs of α and β by maximizing the log-likelihood function using the BFSG quasi-Newton nonlinear optimization method with analytical first derivatives, which is generally regarded as the most reliable nonlinear optimization algorithm (Mittelhammer et al., 2000, p. 199).

Case 2: $\boldsymbol{A} = (1-\rho)\,\boldsymbol{I}_m + \rho\boldsymbol{J}_m$ and $\boldsymbol{\beta} = \beta\mathbf{1}_m$:

we assume that $\boldsymbol{A}$ is an equicorrelated matrix $\boldsymbol{A} = (1-\rho)\,\boldsymbol{I}_m + \rho\boldsymbol{J}_m$ with

$$\delta = 1 - \rho > 0 \quad \text{and} \quad \theta = 1 + (m-1)\,\rho > 0.$$

These parameters restrictions ensure that the matrix $\boldsymbol{A}$ is a positive definite matrix. These conditions with the additional assumptions that the random variables measuring the increase in the crack sizes have same mean and same variance, and have the same tolerance threshold to thermo-mechanical fatigue (TMF) cycle for all the cracks, impose these variables are equicorrelated.
That is, the random variables measuring the crack sizes have the same common variance v_0 and any two of them have also the same covariance v_1, or in others words, these variables are equicorrelated.

Case 2 Continued:

The MLEs $\widehat{\rho}$ and $\widehat{\beta}$ of ρ and β are obtained by simultaneously and iteratively solving the following two equations.

$$0 = \frac{1}{\theta} - \frac{\theta\beta}{g} - \frac{\theta s}{\beta} + 2\theta + (m-1)\frac{1}{n}\sum_{i=1}^{n}\frac{\varphi_1\left(\theta r\left(t_i,\beta\right)\right)}{\Phi_1\left(\theta r\left(t_i,\beta\right)\right)}r\left(t_i,\beta\right), \tag{12}$$

$$\text{and} \quad 0 = -\frac{\theta^2\beta}{g} + \frac{\theta^2 s}{\beta} + \frac{(m-1)\theta}{n}\sum_{i=1}^{n}\frac{\varphi_1\left(\theta r\left(t_i,\beta\right)\right)}{\Phi_1\left(\theta r\left(t_i,\beta\right)\right)}\left(\frac{\beta^{\frac{1}{2}}}{t_i^{\frac{1}{2}}} + \frac{t_i^{\frac{1}{2}}}{\beta^{\frac{1}{2}}}\right) - 1 + \frac{2\beta}{K\left(\beta\right)}, \tag{13}$$

where g, s and $K(\beta)$ are defined in (7), (8) and (10) respectively. It must be pointed out that in this equicorrelated case $0 < \rho < 1$ and $\beta > 0$. Still, $\widehat{\rho}$ and $\widehat{\beta}$ may fall at the boundary, in which case standard asymptotic theory may not be directly applicable. See Self and Liang (1987) for more details on this scenario. So, while solving the above two equations (12) and (13) simultaneously and iteratively, one needs to make sure that $0 < \widehat{\rho} < 1$ and $\widehat{\beta} > 0$. Truncate $\widehat{\rho}$ to 0 or 1, if it is outside this range, as well as truncate $\widehat{\beta}$ to 0 if it is below 0.

Remark:

Note that for $m = 1$, $T \sim EBS\left(\boldsymbol{A}, \boldsymbol{\beta}\right)$ with $\boldsymbol{A} = \left(1 - \rho\right) \boldsymbol{I}_m + \rho \boldsymbol{J}_m$ and $\boldsymbol{\beta} = \beta \mathbf{1}_m$, reduces to $T \sim BS\left(\alpha, \beta\right)$ with $\frac{1}{\alpha} = \theta = 1 + \left(m - 1\right)\rho = 1$. That is, for $m = 1$, $T \sim EBS\left(\boldsymbol{A}, \boldsymbol{\beta}\right)$ reduces to a one parameter BS distribution, $T \sim BS\left(\beta\right)$. Now, when $m = 1$ the equation (13) reduces to

$$\frac{1}{g} - \frac{s}{\beta^2} + \frac{1}{\beta} - \frac{2}{K\left(\beta\right)} = 0,$$

or,

$$\left[K\left(\beta\right) - 2g\right]\beta^2 + gK\left(\beta\right)\beta - gsK\left(\beta\right) = 0. \tag{14}$$

MLE of β can be obtained by solving the above quadratic equation (14). This is the same equation obtained by equating the partial derivative of the log likelihood function of a BS random variable having $\alpha = 1$ with respect to β to zero.

Monte Carlo simulation study for Case 2:

Table 1: Means of $\widehat{\rho}$ and $\widehat{\beta}$ based on Monte Carlo simulation ($\beta = 1.0$)

ρ	n	$m=2$		$m=3$		$m=4$		$m=5$		$m=10$	
		$\widehat{\rho}$	$\widehat{\beta}$	$\widehat{\rho}$	$\widehat{\beta}$	$\widehat{\rho}$	$\widehat{\beta}$	$\widehat{\rho}$	$\widehat{\beta}$	$\widehat{\rho}$	$\widehat{\beta}$
0.3	5	0.5821	0.9889	0.5317	0.9581	0.4966	0.9513	0.4923	0.9427	0.4535	0.9459
	10	0.4630	0.9947	0.4229	0.9787	0.3953	0.9772	0.3905	0.9706	0.3663	0.9730
	20	0.3863	0.9949	0.3621	0.9875	0.3453	0.9855	0.3394	0.9852	0.3289	0.9876
	50	0.3337	0.9966	0.3196	0.9973	0.3153	0.9953	0.3146	0.9929	0.3100	0.9946
	100	0.3167	0.9971	0.3098	0.9978	0.3073	0.9978	0.3070	0.9967	0.3045	0.9980
	200	0.3061	0.9984	0.3039	0.9997	0.3041	0.9981	0.3032	0.9989	0.3021	0.9993
0.5	5	0.7196	0.9986	0.6933	0.9746	0.6966	0.9579	0.6852	0.9595	0.6724	0.9657
	10	0.6450	0.9947	0.6234	0.9836	0.6155	0.9781	0.5997	0.9815	0.5873	0.9825
	20	0.5862	0.9943	0.5638	0.9908	0.5506	0.9909	0.5504	0.9880	0.5416	0.9911
	50	0.5399	0.9981	0.5245	0.9967	0.5218	0.9962	0.5203	0.9940	0.5143	0.9970
	100	0.5182	0.9985	0.5111	0.9974	0.5100	0.9980	0.5095	0.9972	0.5059	0.9988
	200	0.5098	1.0001	0.5062	0.9985	0.5053	0.9985	0.5044	0.9991	0.5032	0.9991
0.7	5	0.8142	1.0122	0.8200	0.9895	0.8278	0.9803	0.8216	0.9822	0.8237	0.9821
	10	0.7837	1.0035	0.7805	0.9947	0.7879	0.9873	0.7822	0.9893	0.7819	0.9883
	20	0.7638	0.9972	0.7532	0.9959	0.7563	0.9918	0.7514	0.9919	0.7506	0.9931
	50	0.7342	0.9964	0.7283	0.9958	0.7266	0.9964	0.7206	0.9969	0.7186	0.9981
	100	0.7183	0.9993	0.7152	0.9974	0.7132	0.9984	0.7106	0.9981	0.7100	0.9987
	200	0.7075	1.0003	0.7077	0.9981	0.7064	0.9989	0.7057	0.9987	0.7056	0.9990

Monte Carlo simulation study for Case 2 Continuation::

Table 2: S.D. of $\widehat{\rho}$ and $\widehat{\beta}$ based on Monte Carlo simulation ($\beta = 1.0$)

ρ	n	$m=2$		$m=3$		$m=4$		$m=5$		$m=10$	
		$\widehat{\rho}$	$\widehat{\beta}$	$\widehat{\rho}$	$\widehat{\beta}$	$\widehat{\rho}$	$\widehat{\beta}$	$\widehat{\rho}$	$\widehat{\beta}$	$\widehat{\rho}$	$\widehat{\beta}$
0.3	5	0.3720	0.2767	0.3138	0.2269	0.2831	0.2077	0.2662	0.1840	0.2262	0.1271
	10	0.3169	0.2001	0.2349	0.1715	0.1946	0.1523	0.1757	0.1371	0.1342	0.0931
	20	0.2368	0.1440	0.1564	0.1233	0.1236	0.1082	0.1081	0.0982	0.0807	0.0675
	50	0.1426	0.0930	0.0912	0.0809	0.0717	0.0691	0.0631	0.0632	0.0458	0.0424
	100	0.0979	0.0646	0.0623	0.0562	0.0489	0.0490	0.0427	0.0447	0.0316	0.0306
	200	0.0694	0.0472	0.0431	0.0404	0.0355	0.0353	0.0298	0.0316	0.0226	0.0217
0.5	5	0.3205	0.2414	0.2808	0.1834	0.2591	0.1439	0.2503	0.1307	0.2308	0.0795
	10	0.2941	0.1740	0.2392	0.1353	0.2139	0.1137	0.2004	0.0996	0.1732	0.0624
	20	0.2410	0.1260	0.1807	0.0982	0.1549	0.0815	0.1410	0.0704	0.1149	0.0445
	50	0.1626	0.0825	0.1099	0.0633	0.0926	0.0534	0.0842	0.0463	0.0657	0.0284
	100	0.1126	0.0568	0.0774	0.0445	0.0647	0.0393	0.0576	0.0318	0.0453	0.0200
	200	0.0795	0.0410	0.0531	0.0317	0.0457	0.0272	0.0402	0.0233	0.0329	0.0147
0.7	5	0.2675	0.2168	0.2271	0.1479	0.2083	0.1126	0.2054	0.0949	0.1889	0.0540
	10	0.2465	0.1536	0.2072	0.1103	0.1888	0.0857	0.1845	0.0737	0.1656	0.0425
	20	0.2129	0.1098	0.1724	0.0803	0.1589	0.0635	0.1499	0.0541	0.1340	0.0321
	50	0.1658	0.0720	0.1254	0.0520	0.1122	0.0422	0.1030	0.0364	0.0876	0.0217
	100	0.1228	0.0509	0.0902	0.0366	0.0780	0.0305	0.0711	0.0253	0.0611	0.0153
	200	0.0891	0.0368	0.0639	0.0270	0.0543	0.0214	0.0512	0.0183	0.0426	0.0109

Hazard function: Cases 1 and 2:

The distribution function F_T in Case 1 is given by

$$F_T(t) = \{1 - [\Phi_1(\kappa r(t))]^m\} I_{\Re_+}(t),$$

and its density function f_T for $t \in \Re_+$ is given by

$$f_T(t) = \frac{d}{dt} F_T(t) = -m\kappa\varphi_1(\kappa r(t)) [\Phi_1(\kappa r(t))]^{m-1} \frac{dr(t)}{dt},$$

The distribution function F_T in Case 2 is given by

$$F_T(t) = \{1 - [\Phi_1(\theta r(t))]^m\} I_{\Re_+}(t),$$

and its density function f_T for $t \in \Re_+$ is given by

$$f_T(t) = -m\theta\varphi_1(\theta r(t)) [\Phi_1(\theta r(t))]^{m-1} \frac{dr(t)}{dt}.$$

In the first case κ is any (fixed constant) positive real number, while in the second case $\theta = 1 + (m-1)\rho$, with $0 < \rho < 1$.
Due to the similarities in functional forms between the distribution functions of Cases 1 and 2, and between their corresponding density functions, we confine ourselves to obtain the hazard function h corresponding to Case 2 only. Now, the hazard function h corresponding to Case 2 is

$$h(t) = \frac{f_T(t)}{1 - F_T(t)} = -m\theta \frac{dr(t)}{dt} \frac{\varphi_1(\theta r(t))}{\Phi_1(\theta r(t))}. \tag{15}$$

Hazard function: Cases 1 and 2 Continued:

Except for the constant m, the expression of the above hazard function (15) is similar to the hazard function of a BS random variable (16) with parameters $\alpha > 0$ and $\beta > 0$, because its distribution function is

$$F_T(t) = \left\{1 - \left[\Phi_1\left(\frac{1}{\alpha} r(t)\right)\right]\right\} I_{\Re_+}(t),$$

and its density function is

$$f_T(t) = -\frac{1}{\alpha}\varphi_1\left(\frac{1}{\alpha} r(t)\right)\frac{dr(t)}{dt},$$

and so the hazard function h of a BS random variable with parameters $\alpha > 0$ and $\beta > 0$ is

$$h(t) = \frac{f_T(t)}{1 - F_T(t)} = -\frac{1}{\alpha}\frac{dr(t)}{dt}\frac{\varphi_1\left(\frac{1}{\alpha} r(t)\right)}{\Phi_1\left(\frac{1}{\alpha} r(t)\right)}. \tag{16}$$

The above hazard function of a BS random variable (16) with parameters $\alpha > 0$ and $\beta > 0$ is identical to the hazard function of an EBS random variable (15) with parameters $\boldsymbol{A} = (1-\rho)\boldsymbol{I}_m + \rho\boldsymbol{J}_m$ and $\boldsymbol{\beta} = \beta\mathbf{1}_m$, for $m = 1$ and $\theta = \frac{1}{\alpha}$.

Remark:

Therefore, all the inferences inferred in Kundu et al. (2008) for the BS hazard function are valid for our EBS hazard functions for Cases 1 and 2. That is, the hazard functions of the EBS distributions corresponding to Cases 1 and 2 are unimodal (of upside down form). The change point, the point where the monotonicity of the hazard function changes from increasing to decreasing, in our EBS hazard functions can be obtained as a solution of a similar non-linear equation as described in Kundu et al. (2008).

Moreover, the different methods proposed by them for estimating the change point will also work for our EBS hazard functions for Cases 1 and 2. As recommended by them the bias-corrected modified moment estimators (BCMME) and approximate BCMME (ABCMME) will perform well, and since ABCMME is a simple explicit estimator, we also suggest its usage for the estimation of change point for our EBS hazard functions for Cases 1 and 2. The derived asymptotic distributions of all the estimators in Kundu et al. (2008) will also work out for our EBS hazard functions for Cases 1 and 2.

Concluding remarks

Thus, with the majority of the remarks in this article we see that our new EBS distribution indeed extends or generalizes the commonly used BS distribution, to model the more typical circumstances of multiple cracks in the fatigue life prediction.

This is the first study of the Birnbaum-Saunders life distribution on multiple cracks, however there is more (yet) to come out on different parameter structures of the EBS distribution. We are currently working on various structures of the shape matrices and the scale vectors of the EBS distribution, and will publish it in a future correspondence.

One can study the model when the number of cracks could vary from individual sample to sample. To accomplish this one has to find the MLE not from a random sample, but from independent EBS random variables each with possible different m_i values. We are currently working on this problem too and will publish it in a future correspondence.

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THANK YOU!