Making sense of negative numbers through metaphorical reasoning

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Abstract

The concept of negative numbers is such an abstract concept that it has been argued that in can only be understood through symbolic reasoning. However, others argue that all mathematical concepts are understood through metaphors. Previous research has identified three important aspects of understanding negative numbers: direction and multitude, proficiency in arithmetic operations and the meaning of the minus sign. This study further explored the theory of conceptual metaphors and metaphorical reasoning by investigating the use of models and metaphorical reasoning when dealing with negative numbers. The data consists of test results from 99 students in the teacher training program and follow-up group interviews The results show that there is a fourth aspect that might create difficulties; relying on metaphorical reasoning using a model that is insufficient. The results also show that metaphorical reasoning is helpful when the student is aware of the limitations of the model.

Introduction and research question

When I ask students why they find negative numbers difficult they often reply that it is because the negative numbers are so abstract and lack connections to the real world. The most common real world connection to negative numbers is the thermometer, at least in Sweden where we use the Celsius scale and have temperatures below zero every winter. A visual representation related to the thermometer is the number line. Reasoning about numbers and calculations in terms of actions on or with these representations is here referred to as metaphorical reasoning. The research question for the study reported her is: How do students make use of these representations and use metaphorical reasoning when solving tasks with negative numbers and how does this relate to their understanding of these numbers?

Previous research concerning negative numbers

Researchers have previously identified three main aspects of understanding the concept of negative numbers. The first aspect is an understanding of the numerical system and the relative size of the numbers (direction and multitude) as well as an understanding of the number zero. (Ball, 1993; Kullberg, 2006; Martínez, 2006). Ball shows that the absolute value aspect (the multitude) of negative numbers is very powerful. These are all different aspects of number sense (Reys & Reys, 1995)

A second aspect is how well the students understand the arithmetic operations (Chacón, 2005; Vlassis, 2004). The big problems when calculating with negative numbers arrive with the subtracting of a negative number (Multiplication is even more problematic but is usually brought in later). Gallardo (1995) showed that it is of great importance if the student understands subtraction only as an operation (taking away) or if they also have a structural understanding (as a comparison between two numbers). On the other hand Linchevski & Williams (1999) did an experimental study where 'subtraction as take away' was understood by the students to be 'the same as adding the opposite number' (through a dice-game and counting on a double abacus) and never used the structural aspect of subtraction. Sfard (1991) indicates that the interiorization of negative numbers is the stage when a person becomes skillful in performing subtractions.

The third identified important aspect is the meaning of the minus sign. The same sign is used both as a sign of operation and as a sign indicating a negative number, that is, indicating the nature of the number (Gallardo, 1995; Kilborn, 1979; Kullberg, 2006, 7-9 Dec.; Vlassis, 2004). In some ways it is unfortunate that the sign is the same, and there has been experimental research where different signs are used for the two different purposes. (Ball, 1993). The different meanings of the minus sign could be described as the operational and the structural aspects of the sign, where the operational meaning is usually introduced long before the structural meaning. Many errors appear when the minus sign indicating a negative number is detached from the number (Herscovics & Linchevski, 1991; Vlassis, 2002).

Many textbooks use visual representations/models[1] (such as the number line, a scale, a time line) and everyday life representations/models (such as temperatures or money) to explain subtraction with negative numbers. According to Linchevski & Williams (1999) some researchers *"…argue against using the existing models for negative numbers […] concludes that the topic of negative numbers should be taught only when the students are ready to cope with intramathematical justifications"* (p. 134). Contrary to this, through their experiment Linchviski & Williams draw the conclusion that at least subtraction with negative numbers can be understood through models; not a single model but a multiplicity of models. Gallardo (1995) suggests teaching negative numbers using discrete models where whole numbers represent objects of an opposing nature rather than using the number line. Kilborn (1979) points out that some teachers use several different models simultaneously during a lesson and that these models seem to confuse the students. Ball (1993), on the other hand, states that no representation capture all aspects of an idea and "*teachers need alternative models to compensate for imperfections and distortions in any given model"* (p. 384). She articulates a dilemma when she asks whether she confuses the children by letting them explore multiple dimensions of negative numbers by introducing different representations.

[1] The word model is here used to be tantamount to a visual or experienced representation. Once seen, this visual representation can be referred to as a mental representation or mental model without being actually visible.

Some work has been done on the use of metaphorical reasoning when dealing with negative numbers (Chiu, 2001; Stacey *et al.*, 2001), which is also the focus of the study presented here.

Metaphorical reasoning.

The theories that have developed around metaphorical reasoning and conceptual metaphors are in some ways an answer to the classical dilemma of how one can learn things about that which one does not know. Learning new things is always about connecting new experiences to already known experiences. Metaphors serve as important links between prior knowledge and new concepts. Metaphors can be defined as understanding one conceptual domain (the target domain) in terms of another conceptual domain (the source domain). Lakoff & Núñes (2000) assert that most mathematical and abstract concepts are conceptualized in concrete terms, for example thinking of numbers as points on a line. There are, they say, two kinds of metaphorical mathematical ideas: grounding metaphors yielding basic ideas and linking metaphors yielding sophisticated or abstract ideas (p. 53). All metaphors have limitations since the target domain is never identical to the source domain. When we construct highly abstract concepts we build up whole metaphorical systems that together characterize the concept. Metaphors make sense of our experiences by providing coherent structure, and thus focusing attention to those aspects of the source and target domains that bear similarities. The theory of metaphorical concepts claims that different metaphors are used to structure different aspects of a concept (Lakoff & Johnson, 1980)

Since the words ‘model’ and ‘metaphor’ are used alternatively by many authors the two concepts are often confused and a clarification is called for. A conceptual metaphor can be described as a set of mappings from a source domain to a target domain (Kövecses, 2002).

mapping

source domain ⟹ target domain

A pedagogical model is a way of representing a mathematical concept and can be used as a conceptual metaphor by mapping the source domain of the model onto the mathematical target domain. Saying “-2+5 is when the temperature is -2º and then goes up 5 degrees” is a way of using the thermometer model as a conceptual metaphor for addition. The model is not a metaphor in itself; it supplies the metaphor with a source domain and can thereby be used as a metaphor. (See also Fischbein (1989 p. 9) for a definition of model). Experiences of temperature change and how that is shown on a thermometer is here the source domain and arithmetic is the target domain. Several different models can have different source domains but similar or even isomorphic mappings to the same target domain if they are being used as metaphors for the same concept. Take for instance a model of an elevator going up and down in a building, temperatures rising and sinking on a thermometer or a glacier growing in winter and melting away in summer. They are all different but when these models are used as conceptual metaphors for arithmetic they have isomorphic mappings to the grounding metaphor 'Arithmetic as Motion Along a Path' (Lakoff & Núñes, 2000). The difference between the models lies in what is moving along the path and what kind of path it is, but the structure is the same.

Some models are already in themselves abstractions and as such removed from a child's everyday knowledge. A model has to be a *model of* something before it can be used as a *model for* something (Gravemeijer, 2005). Understanding of the source domain is a requirement if a model is to function as a conceptual metaphor. "*A metaphor can serve as a vehicle for understanding a concept only by virtue of its experiential basis*" (Lakoff & Johnson, 1980 p.18). Using a particular model as a conceptual metaphor by reasoning about a mathematical concept in terms of this model is what is here referred to as metaphorical reasoning.

Understanding

In *Handbook of Mathematics Teaching and Learning* understanding is defined in terms of connections between ideas, facts and procedures. "…the mathematical idea is understood if its mental representation is part of a network of representations." (Hiebert & Carpenter, 1992 p. 67). An additional condition is of course that this network of representations will help to produce mathematically correct answers to mathematical problems. A very simplified operational definition of understanding negative numbers is to say that it is *both* having the ability to get correct answers when doing different kinds of calculations with negative numbers *and* being certain that the answers are correct. This definition excludes all opinions about which representations, facts or procedures that are essential and it is sufficient for the purpose of this paper.

Method

In this study a test was given to students (n=99) in the teacher training program prior to the topic being dealt with in their mathematics course. The test included calculation tasks on negative numbers, confidence ratings and follow-up questions in order to obtain some mathematical reasoning from the student. There were two groups of students who were enrolled in a one term course in Mathematics Education for young children. They were not obliged to have taken more mathematics than the basic mathematics course (mathematics A). More than 90 % of the students in the two groups were women. The test was given to the students at the end of an ordinary lecture on a randomly chosen day during the course without the students knowing beforehand about the test. The students attending the lecture that day therefore made up the respondent group. Participation in the test was voluntary but all the students chose to respond. The test took 10-15 minutes to complete and was anonymous.

The test results were statistically analyzed and answers to the follow-up questions were categorized according to similarities. As a means of triangulation (Bryman, 2004) three groups of students participated in video recorded group interviews where they discussed their answers and a few related tasks. Each of the three groups consisted of 2-5 students.The students knew me as a teacher in some parts of the course, which might be considered a complication. However, the students had no reason to assume that this test would in any way influence their grades. The test had some open questions where the students were asked to write down their own line of thought in order to get hold of their reasoning. The results of the analysis only concern what the students have written on the test as a reflection of their thinking, not their actual thinking. Data from the group discussion interviews were used as a complement to achieve a more holistic view (Cohen *et al.*, 2000, p.115).

There were three parts on the test. This paper reports on the results of the first part which dealt with subtraction of a negative number. The first question asked only for an answer to a calculation. The second question aimed at measuring the confidence of the student and in the third question the mathematical reasoning was sought.

1a) calculate: (-3) – (-8) = ________

1b) How sure are you that your answer is correct? (choose one answer)

__ very uncertain __ a bit uncertain __ rather confident __ very confident

1c) There are different ways of thinking to reach an answer to a question like this. Try to describe your way of thinking.

Results

As shown in table 1 67 % of the students solved the subtraction task (1a) correctly and 33 % gave an incorrect answer. The most common incorrect answer was -11.

table 1: results from task 1a (percentage and number out of total 99)

answer	5 (correct)	-11 *or* -5	-11 *and* 5	Other / no	Total incorrect
total n= 99	67 % (n=66)	28 % (n=28)	2 % (n=2)	3 % (n=3)	33 % (n=33)

60 % of the students who gave incorrect answers on task 1a chose either ‘rather confident’ or ‘very confident’ on the next question, many of them referring to either a representation like

the thermometer or to arithmetic rules. For those with correct answers the corresponding percentage was 85 %. The answers given to question 1c were categorized as either

- using metaphorical reasoning referring to some mental or visual representation of negative numbers (n=23)
- referring only to an arithmetic rule or a deductive argument (n=71)
- no or irrelevant answer (n=5)

Metaphorical reasoning on the first task included the thermometer, money debts and movements along the number line. The category using metaphorical reasoning (n=23) fell into two distinct subcategories: First; students who referred *only* to metaphorical reasoning on this task (n=14), *all* of which had arrived at a wrong answer, and second; students who used *both* metaphorical reasoning *and* an arithmetic rule (n=9), *all* of which had arrived at a correct answer (see table 2). One person gave two alternative answers; reference to the thermometer rendered the wrong answer and reference to an arithmetic rule rendered the right answer.

table 2: Metaphorical reasoning using temperatures or number line models

Answers of the first subcategory: incorrect answer given (-11)
I think of the thermometer. It was minus 3º and then it got 8º colder.
On a number line or a thermometer I think of where -3 is and keep going -8 to get to -11.
I add -3 and -8. That makes -11. If I have -3 on a thermometer. Take away another 8, that makes -11. [*This student made a drawing of a vertical scale, each number marked from -12 to +5. One mark is on -3 and the text next to it says:*] -3 count down 8 steps to -11
I count up from -8, add 3 which in this case is -3. Think addition. I also have a vague idea that negative numbers sometimes become positive. I would show a thermometer.
It's just automatic. I have simply learned it. Just take a scale with negative numbers and back 8 steps from (-3)

Answers of the second subcategory: correct answer given (5)
I will start by simplifying the expression[2], take away as many brackets and signs as possible: I know that minus and minus makes + (that is in front of and inside the brackets) and then + and – makes – (1:st pair of brackets). Then what remains is just -3+8. (Which I in my head picture on a thermometer-scale and count up 8 steps from -3)
I think that two minus make a plus. I picture a thermometer.
-3+8=5. I see a ladder/temperature scale & "look" where zero is & -3 steps from there and then +8.
From minus three you have to take away minus 8. That makes "+" 8 because it was minus to start off with. [*This student made a drawing of a vertical number line*]
Like my drawing. I remember that two minus cancel each other and makes plus, hence -3+8=5. [*There is a drawing of a horizontal number line*]

[2] In Sweden the word "tal" is used both in the meaning of "number" but also in the meaning of "mathematical expression" or "mathematical task" This can be very confusing. In the translation I will use different words.

As seen in the examples the use of these models are not functional as conceptual metaphors in this task. There is no experience in the source domains of these models that map onto the mathematical operation subtraction of negative numbers (Kilhamn, 2008). The students in the first subcategory, finding no meaning for a double negation in the source domain simply discarded one of the negations as shown in the examples in table 2a. This is elaborated by one student during the interview who had given two different answers: (student A) "*I pictured the thermometer. And on it I took minus 3, I was thinking. First. And then minus minus 8* [emphasizes each word with a movement to the left with her hand] *then it has to be even further away on the thermometer, somehow."* The second subcategory starts off by simplifying the expression using an arithmetic rule (symbolic transformation) until they get an expression that has meaning in the source domain of the thermometer/number line model.

Among the group of students that refer to an arithmetic rule without revealing any metaphorical reasoning, many reveal a transforming of the expression from (-3)-(-8) into -3+8. The line of thought from this simplified expression to the right answer is not expressed. It may be possible that a student is so well acquainted with calculations of this kind that she is not explicitly aware of any thinking here; the answer comes 'automatically'. Student B is an example of this. She finds these calculations easy as she has been working with them a lot, she can "*spit it out*" [indicating with her hand something coming out of her mouth]. She also says that when working with it a lot you "*go further into it*" whereas if you don't work with it it's "*like a foreign language*". However, it may also be that the metaphorical reasoning that takes place is implicit. The number line might be a metaphor the student isn't aware she is using. Student C, who gave a correct answer, writes on the test: "*I thought of the rule – (– make +* ". In the interview when discussing how to simplify the expression -4n-3n she says: "*But it is the same there. You add them together* [makes jumping movements to the left with her hand in the air] *...minus 4. On it. And then you take 3 more then you land on minus 7".* This way of making gestures right and left indicating the use of a number line as an unconscious (tacit) metaphor has also been shown by Edwards (2005). Student D, who only wrote -3+8=5 as an explanation on the test, also suggested that she would use "*a diagram and two points on the minus line"* to explain the solution. Student E has only referred to the rule "*2 minus signs make a plus*" on the written test. When asked during the interview if she also used the number line she replied: "*No! I just thought minus minus…make…plus. I counted minus three plus eight. Well, I guess I assumed a number line in a way. No, I thought of .. , I pictured a thermometer!".*

The students who arrive at an incorrect answer after referring to an arithmetic rule (n=14) go on with symbolic reasoning, trying to remember what to do with the minus signs. They try to make sensible use of the signs and rules without having a metaphor to refer to. (see table 3) . Errors identified in previous research, such as ‘adding before subtracting’ and ‘detachment from the minus sign’ (Herscovics & Linchevski, 1991; Vlassis, 2002), are frequent. These errors are clearly connected with the minus sign and subtraction as an operation (two of the identified aspects).

Table 3: Examples of incorrect answers given with only reference to an arithmetical rule

- is equal to + (8+3=11) *(gives the answer -11)*
When there are two minuses next to it makes plus., then there is one minus left that you save. *(gives the answer -11)*
Two minus make a plus, or else the brackets have something to do with separating the minus sign and it becomes more minus. *(gives two answers; 5 or -11, believes -11 is correct)*
I know that two minus make plus. The minus sign in the middle that connects the numbers make the expression negative. *(gives the answer -11)*
8+3=11 and remember that it was a negative number.[3] *(gives the answer -11)*
When it is –(-8) it is even more minus *(gives the answer -11)*
First I thought -5 because I took -3+8 but then I saw that it was -8 also & I realized that I shouldn’t mix in a + of course. *(gives the answer -11)*
I don’t know how to do it, I need more time. I seem to remember that same sign like in this case makes a positive number. That far I’m in. *(gives no answer at all)*

The results and different categories can be summarised as in a figure 1.

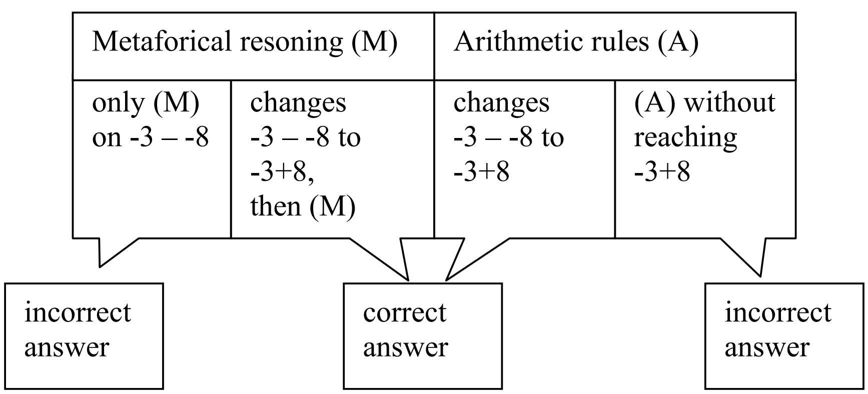

Figure 1: overview of categories

[3] The student uses the words ‘negativt tal’ which could mean negative number or negative expression.

Discussion

It is interesting to note that many of the students who failed the calculation task neither seemed to understand these numbers well enough to solve the task, nor did they anticipate that they might be wrong. They seemed to believe that the metaphorical reasoning they used would give them a correct answer.

In this data the largest group of incorrect answers (n=14 out of 33) was found to be those who *only* used metaphorical reasoning, and most of them declared that they were rather confident (n=6) or very confident (n=3) about their answer being correct. Another group of students (n=9) used metaphorical reasoning as a complement to symbolic transformation (changing –(-8) into +8). These students *all* arrived at a correct answer. It seems as if metaphorical reasoning is only helpful when it is used with understanding, which is, when the student is aware of the constraints of the metaphor and is capable of treating numbers as entities without meaning in order to transform them into something that carries meaning and has similarities to the representation at hand. It is crucial that students become aware of the constraints of the representations and models that underlie their understanding. This is in line with the results of Chiu (2001), claiming that experts know the limitations of the metaphors they use and therefore learn when to use each metaphor.

The largest group of correct answers were found among students who transformed the operation (-3)-(-8) into -3+8 and then calculated the answer. It is possible, as seen in the group interviews, that these students implicitly used a mental number line, scale or thermometer or other representation when dealing with the operation -3+8. Those who explicitly did so arrived at the correct answer. Students who gave a wrong answer without referring to metaphorical reasoning gave more arguments about the number of minus signs or gave plus priority over minus and in general got all tangled up in arithmetic rules without meaning. A possible interpretation of this data is that metaphorical reasoning is essential in order to create meaning in the calculation and judge if the answer makes sense or not. This argument supports Chiu's (2001) claim that novices more often than experts use metaphorical reasoning to verify their results. An expert would have done so many similar calculations that she need not verify it anymore (as expressed by student B) whereas a novice would need some way of justification. In this respect the metaphor serves as scaffolding (Vygotskij, 1999). In addition to the three previously shown aspects of understanding negative numbers these results shed light on a fourth aspect that might create difficulties and cause incorrect calculations; relying on metaphorical reasoning using a model that is insufficient. Maybe that is a sign of a very poorly developed net of mental representations. The results of this study suggest that teachers should make use of the grounding metaphors and experience based models but emphasize their limitations concerning negative numbers. Knowing the potentials and constraints of a model is necessary if it is to function as a conceptual metaphor and for the learner to be creative in striving to understand. As a contribution to the research society theses results suggest that the debate should not concern which model to use and why one model is better than another but rather what are the consequences of our use of metaphors and how do we deal with these consequences.

References

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal*(93), 379-397.

Bryman, A. (2004). *Social research methods* (2nd ed.). Oxford: Oxford University Press.

Chacón, A. M. A. (2005). Difficulties found by the students during the study of subtraction of integer numbers. In M. Bosch (Ed.), *Fourth Congress of the European Society for Research in Mathematics Education (CERME 4)* (pp. 643-651). Spain: http:/ermeweb.free.fr/CERME4/.

Chiu, M. M. (2001). Using metaphors to understand and solve arithmetic problems: Novices and experts working with negative numbers. *Mathematical thinking and learning, 3(2-3)*, 93-124.

Cohen, L., Manion, L., & Morrison, K. (2000). *Research methods in education* (5th ed.). London: Routledge Falmer.

Edwards, L. D. (2005). Metaphors and gestures in fraction talk. In M. Bosch (Ed.), *Fourth Congress of the European Society for Research in Mathematics Education (CERME 4).* (pp. 92-101). Spain: http:/ermeweb.free.fr/CERME4/.

Fischbein, E. (1989). Tacit models and mathematical reasoning. *For the learning of mathematics, 9*(2), 9-14.

Gallardo, A. (1995). Negative numbers in the teaching of arithmetic. In D. Owens, M. Reeds & G. Millsaps (Eds.), *Proceedings of the 17th Annual Meeting for the Psychology of Mathematics Education (North America Chapter)* (Vol. 1, pp. 158-163). Ohio, USA: ERIC Clearinghouse for Science, Mathematics and Environmental Education.

Gravemeijer, K. (2005). What makes mathematics so difficult, and what can we do about it? *Proceeding of the International Meeting in honour of Paolo Abrantes. Mathematics education: paths and crossroads.* (pp. 83-101). Lisboa, Portugal: Faculdade de Ciências da Universidade de Lisboa.

Herscovics, N., & Linchevski, L. (1991). Pre-algebraic thinking: Range of equations and informal solution processes used by seventh graders prior to any instruction, *Proceeding of the 15th Conference of the International Group for Psychology of Mathematics Education (PME 15)* (Vol. 2, pp. 173-180). Assissi, Italy: PME.

Hiebert, J., & Carpenter, T. (1992). Learning and teaching with understanding. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). USA: Macmillan Publishing Company.

Kilborn, W. (1979). *Ämnesmetodiska processanalyser i matematik inom komvux* (No. 8). Stockholm: Högskolan för lärarutbildning. Institutionen för pedagogik. Forskningsgruppen för vuxenpedagogik och återkommande utbildning.

Kilhamn, C. (2008). Addition and subtraction of negative numbers using extensions of the metaphor "arithmetic is motion along a path": Manuscript accepted for presentation at NORMA 08, Copenhagen, Denmark (April).

Kullberg, A. (2006, 7-9 Dec.). From learning study to design study, *Paper presented at EARLI SIG 9 Biennial Workshop*. University of Hong Kong.

Kövecses, Z. (2002). *Metaphor, a practical introduction*. New York: Oxford University Press.
Lakoff, G., & Johnson, M. (1980). *Metaphors we live by.* Chicago: University of Chicago Press.
Lakoff, G., & Núñes, R. E. (2000). *Where mathematics comes from*. New York: Basic Books.
Linchevski, L., & Williams, J. (1999). Using intuition from everyday life in 'filling' the gap in childrens extension of their number concept to include negative numbers. *Educational Studies in Mathematics, 39*, 133-147.
Martínez, A. A. (2006). *Negative math: How mathematical rules can be positively bent.* Princetown, NJ: Princetown University Press.
Reys, B. J., & Reys, R. E. (1995). Perspektiv på number sense och taluppfattning. *Nämnaren, 22*(1), 28-33.
Sfard, A. (1991). On the dual nature of mathematical conceptions. *Educational Studies in Mathematics, 22*(1), 1-36.
Stacey, K., Helme, S., & Steinle, V. (2001). Confusion between decimals, fractions and negative numbers: A consequence of the mirror as a conceptual metaphor in three different ways. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th International Group for the Psychology in Mathematics (PME 25)* (Vol. 4, pp. 217-224): Freudenthal Institute, Utrecht University.
Vlassis, J. (2002). The balance model: Hindrance or support for the solving of linear equations with one unknown. *Educational Studies in Mathematics, 49*, 341-359.
Vlassis, J. (2004). Making sense of the minus sign or becoming flexible in `negativity´. *Learning and Instruction, 14*, 469-484.
Vygotskij, L. S. (1999). *Tänkande och språk* (K. Öberg Lindsten, Trans.). Göteborg: Bokförlaget Daidolos AB. Original work published in 1934.