CHILDREN'S EARLY WORK WITH DIVISION

Frode Rønning

Sør-Trøndelag University College, Trondheim, Norway

The purpose of this paper is to discuss the way very young children handle problems connected to division. This discussion is based on classroom observations and it is linked to a discussion of how Norwegian textbooks present the first encounter with division. In textbooks division is often simply regarded as reverse multiplication. Based on the classroom I will argue that it could be worthwhile pursuing division as a process in its own right and postpone the strong link to multiplication until later.

INTRODUCTION

Following a social constructivist view on learning (Ernest, 1998) I take the stance that knowledge is developed by the learner in interplay with teachers, fellow learners and the teaching material. A constructivist view entails that knowledge is not detached from the knower but actively built up in a process where the learner organises his/her experiences in relation to previous knowledge (von Glasersfeld, 1995). When it comes to teaching, this means that it is of vital importance to establish strong links between the children's own ways of thinking and the input provided by the teacher and the teaching material. Along with the development of a basic number concept children will also develop ways of solving problems involving numbers, both in everyday life and in school contexts. Algorithms that children develop for solving "school problems" is referred to by Steffe (1994) as child generated algorithms. These algorithms usually differ from the standard algorithms taught in school and are also often quite inefficient and insufficient. Nevertheless they are important, and for the teacher the challenge will be to create links between the child generated algorithms and the standard algorithms.

As Fischbein, Deri, Nello, and Marino (1985) write there seems to be certain natural, intuitive models for multiplication and division. The intuitive multiplication model is based on equal grouping and is often referred to as repeated addition. For division the basic model is that of sharing equally between a given number of persons, referred to as partitive division. Although the equal grouping model for addition and the partitive model for division will be insufficient for modelling situations that the students encounter at a later stage, they have a strong influence on how students perceive the conceptual field of multiplication and division.

I will discuss the need for working with more varied models in the early work with multiplication and division. An examination of current Norwegian textbooks indicates that the main approach to division is through the partitive model, and the quotitive model is more or less tacitly introduced, if it is at all mentioned. In addition the study of the textbooks indicates that division already in the very beginning is presented as the reverse operation of multiplication. Based on observations of children at an early stage of working with division problems I will discuss whether it might be desirable to a larger extent to work with division as an operation in its own right, and I will discuss what considerations should be made when the link between division and multiplication is introduced. This could be formulated as the following research question: *What are children's intuitive ways of approaching division problems, and how do these approaches fit with the way division is introduced in textbooks?*

THEORETICAL FRAMEWORK

Children's first encounter with multiplication is usually connected to situations involving a number of groups of objects having the same number in each group (Greer, 1992). Later other models for multiplication will appear but the equal grouping model (repeated addition) is recognised as the basic intuitive model for multiplication (Fischbein et al., 1985). Fischbein et al. discuss two models for division, partitive (or sharing) division and quotitive (or measurement) division but they claim that "*there is only one intuitive primitive model for division problems* – the partitive model" (p. 14, emphasis in original) and that the quotitive model is acquired later as a result of instruction. According to Fischbein et al. these intuitive models for multiplication and division reflect the way the concept was initially taught at school and they correspond to features of human mental behaviour that are primary, natural and basic.

Following Steffe (1994, p. 7) a child's solution process can be described in three steps. First there is an experiential situation as perceived by the child. Then there is the child's procedure to deal with the situation, and finally there is the result. These steps may involve different challenges. To get from the situation to the procedure the child has to determine which arithmetic operation should be used. In this step familiarity with a diversity of models for each arithmetic operation is expected to be helpful. After the operation has been decided the task is to carry out the actual calculations, and this step may be more independent of the chosen models for the situation. Here the child may use different strategies irrespective of how the actual situation is modelled. In early learning these steps are not clearly separated.

Neuman (1999) has studied small children's work with division problems. She distinguishes between the situational aspect, meaning the modelling of the practical situation, and the computational aspect, meaning the procedure to obtain the answer. These two aspects correspond to Steffe's (1994) three steps in the sense that the situational aspect corresponds to the experiential situation and the computational aspect corresponds to the procedure and the result. When the children are dealing out one at a time there is hardly any difference between the situational and the computational aspect. However, Neuman found that when the children use repeated addition to build up the dividend the computation is more akin to a quotitive model although the situation is described as partitive. One example to show this is the following. The problem is to share 28 marbles equally between 7 boys, and one pupil explains the answer 'four' by saying: "First I do … seven and seven, that's fourteen … then fourteen and fourteen's twenty-eight" (Neuman, p. 114). The pupil measures how many times seven goes into 28, whereas the situation asks for the number which goes 7 times into 28.

The most basic strategy for division is dealing out objects one at a time by direct counting, and Mulligan and Mitchelmore (1997) write that direct counting "achieve the aim of creating equal-sized groups, but the calculation procedure does not reflect this structure" (p. 318). Repeated subtraction can be applied directly in quotitive division where the required number in each group is taken away repeatedly until the initial set is empty. In partitive division however, the number in each group has to be guessed, or if the process is carried out by sharing out one item at a time the required number is seen only when the process is completed. Vergnaud (1983) discusses two ways to look at multiplication as a unary operation – the scalar and the function operator – using the example "if the price of one cake is *a*, what is the cost of *b* cakes?" Here *b* is called the scalar operator and *a* is called the function operator. In this language partitive division amounts to reversing the scalar operator, and quotitive division amounts to reversing the function operator.

The most advanced division procedure is when the solution is guessed and checked by multiplication. This procedure is based on the understanding of multiplication and division as inverse operations. Anghileri (2001), in a discussion of partitive and quotitive division describes quotitive division as "an equally powerful interpretation" (p. 86). I will argue that in some respects the quotitive model may be even more powerful than the partitive model, in particular in order to understand division as the inverse operation of multiplication. A model which is quite different from these models is the splitting model suggested by Confrey (1994). This model is based on a repeated halving procedure, and Confrey suggests that splitting is taken as a primitive action complementary to counting. The splitting model exhibits a structure like exponential growth.

METHOD

I will discuss three episodes that are based on observations made in a class consisting of 18 1st grade pupils (6-7 years old) in a small English school. I was an observer in this class in all of the mathematics lessons for three weeks. In the whole class teaching I was passively observing and after having been in the class a few times, I video taped the sessions. When the pupils were working in groups or individually I interacted with them in ways that a teacher would do, and I video and/or audio taped also some of these sessions. I had no influence over the topics the class worked with, or the ways in which the topics were handled. However, the teacher said that she made some adaptations due to my presence. The episodes that are included in this paper are based on video and audio recordings. All three episodes take place on the same day but in different classroom settings, partly in a whole class situation and partly when the children are working in small groups.

Furthermore I will discuss the relation between multiplication and division by analysing excerpts from three Norwegian textbooks on the level where division is first introduced as a formal arithmetic operation. The textbooks that are chosen are all recent editions adapted to the current Norwegian national curriculum.

THE CLASSROOM EPISODES

Episode 1

This is a whole class situation and the class is working with halving. First they have been rehearsing with numbers by saying out statements like "half of eight is four". Afterwards the teacher picks out 10 plastic cubes from a box and asks two children to come to the board. Following the earlier rehearsing they establish that "half of ten is five". The teacher then shares out the cubes in a "one for you – one for you" manner and when all the cubes are shared out, the two children count their cubes and confirm that they have five each. This lays the foundation for the perception of division as 'sharing equally' and that sharing between two is the same as halving. Next the teacher picks out eight plastic cubes from the box and asks two children to come to the board. A third child is given the cubes and is asked to share the cubes equally between the two others. After the sharing process is completed the two children find that they have four cubes each and the teacher prepares the following sentence on the board

___ shared equally between ___ is ____

Here the children are supposed to suggest what should be filled into the blank fields. Below the teacher writes the same with mathematical symbols ($8 \div 2 = 4$) and says out while writing: "Eight divided by two is four". The teacher states that the sign ÷ is "another way of saying divide by or share between". After another example of the same kind the situation is changed into sharing between five children. First, Kyle, who is sharing out, is given ten cubes. He gives one cube at a time to each of the five children until there are no more cubes left. Each child now has two cubes and 10 ÷ 5 = 2 is written on the board. Now Kyle gets 20 cubes to share between the same five children and the following conversation takes place in order to fill in the answer in 20 ÷ 5 = __.

Teacher: Does anyone think they know the answer already?

Amy: Four.

Teacher: Why do you think the answer is going to be four?

Amy: When they were going round with ten they had two, and then they just had to double that.

In the excerpt of the dialogue above Amy shows that she is able to find the answer by generalising. She knows that 10 divided by 5 is 2, and from this she infers that 20 divided by 5 must be 4. It seems that she is seeing a general pattern in the sense that when the amount to be shared is doubled, the outcome for each person will also be doubled. I support this claim by the fact that she is actually using the word 'double' in her utterance, hence she is not just going from 10 to 20 but from 10 to "double 10".

Episode 2

Here the children are sitting in groups of four and they are playing a game in pairs that involves solving division tasks. The tasks are given with numbers and symbols only, for example 8 ÷ 4 = __ . They have centicubes available, and a calculator that they use to check their answers. At the bottom of the task sheet is written "Knowing division facts up to 25 ÷ 5". Amy and Alice are playing together and when they get to 20 ÷ 4 they are stuck. They sit for some time without coming up with any suggestion about how to solve the task and I decide to intervene. I suggest that they leave that particular task for a while and instead try with some other numbers. I ask "what is eight divided by two". The answer "four" comes immediately. Then I ask "what is eight divided by four". Again the answer comes immediately, "two". I continue to ask "what is twelve divided by two". The answer "six" comes without hesitating. I ask "what is twelve divided by four" and there is no answer.

In Episode 1 Amy observed that if the amount to be shared was doubled the outcome would be doubled. Here I had hoped that she or Alice would make the observation that if the number to share between was doubled the outcome would be halved. They do not seem to make that observation. I continue by inventing a story and the following conversation takes place.

Teacher:	Imagine that you have twelve apples and you are going to share between you and Amy, then you would get six each … right? You just said that. But what if you had the same twelve apples and also Jack and Jamie [sitting opposite them at the table] should have a share.
Alice:	We would share all.
Teacher:	Yes, and what would happen with your lot of apples then?
Amy:	I would give three of mine to Jamie and she would give three of hers to Jack.
Teacher:	Then you have done twelve shared out between how many?
Amy:	Four.
Teacher:	And how many are you left with?
Alice:	Three.
Teacher:	Now imagine that you have twenty apples and you are going to share between yourself and Amy. If you have twenty apples.
Alice:	So it's ten each.
Teacher:	Ten each.
Amy:	Five each, between all of us.

In this excerpt the girls, probably supported by my way of presenting the story, develop a way of dividing by four by successive halving. Amy's last utterance indicates that she is not in need of saying out for herself the intermediate result after dividing by two. This is said, however, by Alice and reinforced by me, and then Amy very suddenly states the final result "five each, between all of us". The model that is used here is the splitting model (Confrey, 1994).

Episode 3

In the last episode Amy wants to solve 15 divided by 5. She cannot come up with an immediate answer and after a short while of thinking she says "I'm going to use the cubes". She then counts 15 cubes which she distributes one by one into five heaps. When she has finished she counts the cubes in each heap and states "Fifteen divided by five is three".

In this case none of the strategies that she had developed and used earlier in the lesson were applicable so she falls back on the basic strategy of sharing out one by one and then counting the result.

TEXTBOOK PRESENTATIONS OF DIVISION

In school, multiplication is usually presented and worked with before division but problems involving multiplication and division can be and are solved by children before they have been exposed to any formal teaching on the subject (Mulligan & Mitchelmore, 1997; Neuman, 1991). In a Norwegian textbook for 4th grade (Solem, Jakobson, & Marand, 2006) the first formal encounter of division is through the following problems (p. 5[1]).

> Three children eat two pizza slices each. How many do they eat altogether?
> $3 \cdot 2 =$ __ . They eat __ pizza slices altogether.
> There are six sweets left. Tom and Eric share them. How many do they get each?
> $6 : 2 =$ __. They get __ sweets each.

At the bottom of the same page is written: “Multiplication and division are inverse arithmetic operations. $6 : 2 = 3$ and $3 \cdot 2 = 6$.”

The first situation in this example can be interpreted as $2 + 2 + 2 = 6$ whereas the second situation is $3 + 3 = 6$, or $2 \cdot 3 = 6$. The interpretation $2 + 2 + 2 = 6$ is meaningless in the situation with the sweets. This example is similar to an example discussed by Vergnaud where he comments that “[the situations] are not conceptually the same, although because of the commutativity of multiplication they may be mathematically equivalent” (Vergnaud, 1988, p. 146). If, however, the division example had been described by a quotitive model it would have been modelled by $3 \cdot 2 = 2 + 2 + 2$, and the reversibility could have been observed without having to invoke commutativity of multiplication.

In another book (Alseth, Kirkegaard, Nordberg, & Røsseland, 2006) the authors seem to be conscious about different situations giving rise to partitive and quotitive division. On page 104, under the heading “We practice division”, we see three monkeys holding four bananas each. Under the picture is written “We write $12 : 3 = 4$”. On the side of the picture we see two creatures. One of them says “Now I have shared equally. That gave 4 to each”. The other one says “Yes, that is correct because $4 \cdot 3 = 12$”. On page 105 we find the text “20 carrots are shared out between a number of zebras. Each zebra gets 5 carrots. How many zebras will get carrots?” We see a picture with four bunches of carrots, five carrots in each, and the two creatures saying: “Look, it is enough for four zebras. That is correct because $4 \cdot 5 = 20$”.

In this book the situational aspect (Neuman, 1999) is clearly different in the two examples. In the example with the monkeys a partitive situation is modelled, and in the example with the zebras a quotitive situation is modelled. However, when the answer is tested using multiplication both examples use a model where the dividend is measured with the divisor. In the example with the

[1] All the quotations from textbooks are originally in Norwegian and translated by me.

monkeys this could give the same confusion as Neuman observed with the problem "28 marbles shared equally between 7 boys", and the pupil measures how many 7's are there in 28.

In a third textbook division is introduced through the example 15 : 5 = 3. In the example the number 5 is explained as "the number of parts 15 is divided into". Below this text is an illustration of 12 apples to be shared equally between three children where the 12 apples are separated into three equal sets, with four apples in each set (Bakke & Bakke, 2006, p. 106). A little later in the book the following problems appear, the first problem showing a partitive situation and the second a quotitive.

1. A chocolate is partitioned in 30 squares. 5 persons are sharing the chocolate. How many squares does each person get?
2. We have 48 buns. They are going to be packed in bags with six buns in each bag. How many bags do we need? (p. 113)

It seems to be taken for granted that the solution to the second problem is given by 48 : 6 although this contradicts the earlier convention that the divisor represents the number of parts.

DISCUSSION

In the development of multiplicative thinking there is a goal that children should see multiplication and division as inverse operations. It could be argued that this is advantageous for both their conceptual and procedural knowledge (Hiebert & Lefevre, 1986) of multiplication and division. In terms of conceptual knowledge the argument is that pupils should develop the understanding that dividing by a is the same as multiplying with $1/a$ thereby developing understanding for the multiplicative inverse. In terms of procedural knowledge it will make computations more efficient if one can solve division problems by evoking knowledge about the multiplication table, in particular when doing long division. I will argue however, that in children's first encounter with division, important conceptual development may be lost if the children are not given time to investigate division as a process in its own right. In children's first encounter with the concept of division, which commonly is in situations involving sharing equally, the multiplicative structure is not readily apparent. As the classroom examples in this paper, and also a number of other studies, show, children treat division as an independent process without linking it to multiplication. Despite the fact that the class where I made my observations recently worked with multiplication there is no sign of employing multiplication facts in solving the problems with division. This is coherent with the findings of Neuman (1991, 1999) who worked with children of about the same age as I did. Commenting on the view on division as reverse multiplication Marton and Neuman (1996) write

“[I]t was not with division of this type that most of the children in Neuman’s investigation addressed the problems (p. 319).

The examples that I have referred to seem to indicate that when children move beyond the most basic ‘sharing out one by one procedure’ they might be more prone to venture into procedures like splitting or derived fact strategies building on more basic division facts than linking the division process to multiplication. However, the textbooks I have examined introduce the link between division and multiplication at a very early stage, and merely based on experiences with partitive division.

I will argue that to obtain a good basis for developing understanding for the link between these two processes it is desirable to give the children more experience with models involving quotative division. In my view this model gives a more explicit experience of a multiplicative structure because the process of repeatedly taking away is more directly embedded in this model than in partitive division.

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