

DIFFERENT VIEWS – TEACHER AND ENGINEERING STUDENTS ON THE CONCEPT OF FUNCTION

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This study analyses what kind of conceptions teacher students and engineering students have about the function concept, and how these conceptions differ between the two groups. The study was conducted through questionnaires, and 34 students at a Swedish university participated. The function concepts of the students have been classified according to modified versions of models presented by Vinner & Dreyfus, Sfard and DeMarois & Tall. The study shows that the students primarily have operational concepts of function, with only a couple of students having a structural function concept. The study also shows distinct differences between prospective compulsory school teachers and engineering students, where the former have less developed functional concepts.

THEORETICAL FRAMEWORK

Different approaches have been developed to explain the mechanisms governing concept acquisition. For example, in mathematics education there has been considerable discussion concerning the distinction between *concept definition* and *concept image*, the concept definition being the formal mathematical definition, while the concept image is a much wider concept, representing “*the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes.*” (Tall & Vinner 1981, p. 152).

Regarding the concepts themselves, Sfard (1991, 1992) speaks of the duality of mathematical concepts, in that they can be regarded both as *processes* and as *objects*. While “*there is a deep ontological gap between operational and structural conceptions*” (Sfard 1991, p. 4), the two are not mutually exclusive, but rather complementary. Sfard (1991) has also formulated an influential theory of concept formation. According to this model, concept formation consists of three consecutive stages: *interiorization*, where you get acquainted with the processes behind the concept by performing operations on already familiar mathematical objects; *condensation*, where you get more familiar with the concept, gaining increasing capability to switch between different representations of it; and *reification*, where you gain the ability to view the concept as an object in its own right. For Sfard, this last step is qualitatively different from the first two.

Previous results on the concept of function

The process of reification is by no means an easy one. Several studies (e.g. Hansson 2006; Norman 1992; Sfard 1992; Even 1990; Vinner & Dreyfus 1989) show that even students who have come far

in their studies, and in some cases practicing teachers, do not have a reified concept of function, but rather a process-oriented view. Sfard claims that this needs to have consequences for the teaching of mathematics. One should not introduce structural descriptions of concepts until they are needed, and, more specifically, one should never introduce new concepts in structural terms (Sfard 1992).

When students first encounter the definition of a concept, more often than not they already have concept images, which may be more or less developed. Of course the concept definition will influence the concept image, but when the concept is used in practice, it is almost always the concept image which is evoked. (See e.g. Attorps 2006; Hansson 2006.) Earlier studies (e.g. Akkoç & Tall 2002; Tall & Bakar 1991) indicate that prototypes, that is, standard examples of the concept used for a pedagogical purpose, tend to contribute strongly to the concept image, even though they are often chosen in order to highlight just one particular aspect of the concept. Hence different aspects of the concept image may very well be contradictory, since different aspects of the concept image are used in different contexts. This is called *compartmentalization*, and has been detected in several studies (e.g. Eisenberg 1992; Vinner 1992; Vinner & Dreyfus 1989). Moreover, many studies (e.g. Akkoç & Tall 2002; Meel 2000; Vinner & Dreyfus 1989) have shown considerable discrepancies between students' concept definitions and concept images.

Research questions

Although students' understanding of the function concept has been studied by quite a number of researchers internationally (e.g. Akkoç & Tall 2002; Meel 2000; Even 1993; Tall & Bakar 1991; Vinner & Dreyfus 1989), not that much research on the subject has been done in Sweden. Therefore, one of the aims of this study is to investigate what the participating students' conceptions of the function concept look like, and, if possible, to compare this with the results of studies conducted elsewhere. Hence, the first research question posed in this study is: What is the students' understanding of the function concept? More specifically: How do the students define the concept of function? What do their concept images for the function concept look like?

In my opinion a good conceptual understanding of mathematics is of special importance for would-be teachers. Therefore I have chosen to conduct my study on a group of teacher students, and to compare these students with a group of engineering students. These students also study quite a lot of mathematics, but their goals are different, and more aimed at the use of mathematics in a practical setting. The second research question posed in this study is: What are the differences in the understanding of the function concept between the teacher students and engineering students in the study?

METHODOLOGY

The study was conducted at a Swedish university, and the participants were teacher students currently attending a course in calculus (14 students), and first-semester 5-year engineering students, also attending a course in calculus (20 students). The teacher students had taken more than one semester of mathematics (except for the 3 students aiming at upper secondary school, who had only taken a course in algebra), while the engineering students had only taken a course in algebra.

The data were gathered by questionnaires. The students were asked to associate freely regarding the concept of function, and to construct a “mind map”. They were then presented with a number of mathematical expressions and figures, and were asked to determine which of these represented functions, and to rate the degree of certainty of their answers. Furthermore, they were asked for their opinion on the possibility of constructing a function with certain given characteristics, and finally they were asked to state their own definition of the concept of function. When classifying the answers, use has been made of categorizations presented by Vinner & Dreyfus (1989), Sfard (1991) and DeMarois & Tall (1996).

RESULTS

The first research question deals firstly with the students’ definitions of the function concept. Classifying the definitions according to a modified version of a categorization developed by Vinner & Dreyfus (1989), it was found that most students gave definitions which could be described as process-oriented, and that only a small minority gave structural definitions. Furthermore, nearly a third of the students failed to provide any meaningful definition whatsoever.

The classification makes use of eight categories, of which category 3 is not used by Vinner & Dreyfus. One is a “no answer”-category, and the order of the other seven categories more or less traces the historical development of the function concept (see for instance Kleiner, 1989). The categories are the following (each followed by an example from the questionnaires):

- 1. Correspondence.** A function is any correspondence between two sets that assigns to each element in the first set exactly one element in the other set.

“A function always gives just one value when you insert a value. If you have one set which is the domain and insert one of those values into the function you get one of the values in the range.”
(T 2)¹

¹ The participants are identified by a letter and a number. The letter describes the category: T for teacher, E for engineer. All quotes have been translated from Swedish by the author.

2. Dependence relation. A function is a dependence relation between two variables.

“A function depends on a variable. Depending on what value the variable has you get a unique value of the function.” (E 11)

3. Machine. A function is a “machine” that transforms variables (which need not be numbers) into new variables. In this case no explicit mention of domain and range is made.

“A ‘machine’ which to any input-variable assigns a specific number or something similar.” (E5)

4. Rule. A function is a rule. The difference from **3.** is that a regular behaviour is expected, whereas the machine could conceivably perform totally different transformations of different elements.

“A description of a pattern, which varies depending on different variables.” (E 7)

5. Operation. A function is an operation or manipulation. Here the input values are assumed to be numbers, on which mathematical operations are performed to yield the output value.

“A set of operations giving the same result if you insert the same value.” (E 17)

6. Formula. A function is a formula, an algebraic expression or an equation.

“A function is a formula for which value y assumes for any given value of x .” (T 1)

7. Representation. The function is identified, in a possibly meaningless way, with one of its representations.

“A curve where one x -value has one y -value.” (T 3)

8. No answer or a meaningless answer.

“A function is an explanation of how something works.” (E 4)

It is worth noting here, that the definition given in the textbook used by the teacher students (Rodhe & Sigstam 2000, p. 88) is of category 2, while the textbook used by the engineering students (Adams 2006, p. 24) gives a definition of type 4 (but with explicit mention of domain and range).

Table 1. The number of students’ answers in the eight categories

Category	1	2	3	4	5	6	7	8
Number of students	1	1	6	2	8	6	6	4

Two students fall into categories 1 or 2, the categories that resemble the structural definition of function, while 10 students end up in categories 7 or 8, failing to give a useful definition.

The second part of the first research question concerns the students' concept images for the function concept. According to Even (1990) the essential features of the concept of function in the modern sense are *arbitrariness* and *univalence*. Arbitrariness means that the value of a function at any given point is independent of the value at other points, but also that the domain and range can be arbitrary sets; specifically they need not be sets of numbers. Univalence simply means that for each element in the domain there is a unique element in the range.

In classifying the students' conceptions of the function concept, a model constructed using elements from the classifications of Sfard (1991) and DeMarois & Tall (1996) has been used. The students' conceptions of the function concept have been ordered into *pre-operational*, *operational* and *structural* conceptions. A student with a pre-operational conception has a rudimentary and inconsistent concept image. A student's conception of function is operational if she clearly views a function as a process, and structural if she is also able to view the function as an object in its own right. Using this classification it was found that 12 students had pre-operational and 20 students operational conceptions of the function concept. Two students had something resembling a structural conception. In the following, some interesting aspects of the answers to the questionnaire, and their implications regarding the concept images of the students, will be noted.

The most common concept to appear in the mind maps (20 students) was the concept of graph or curve. Yet only one student mentions the vertical line test. Common are also such calculus concepts as derivative and integral, as well as the function machine and terms like formula, expression and operation. As for the essential features mentioned above, 8 students mention domain/range, and 4 mention univalence. Notable by their absence are such concepts as inverse function and composite function, as well as examples of standard functions. Only a handful of students mention any of these concepts in their maps. The students were also asked to determine whether a number of expressions and graphs could be said to represent y as a function of x . Some of these expressions, together with the distribution of Yes and No answers, are presented in Table 2 below.

Table 2. The distribution of students' Yes and No answers concerning certain of the expressions

Expression	$x^2 + y^2 = 4$	$xy^2 = 5$	$x = 3$	$y = 3$	$f(x) = 3$	$y = \begin{cases} -3 & x < 0 \\ e^x & x \geq 0 \end{cases}$	$y = \begin{cases} 1 & x \text{ rat.} \\ 0 & x \text{ irr.} \end{cases}$
Yes	24	23	4	13	22	32	24
No	10	10	29	20	12	2	10

We see that a majority of the students consider both of the first two expressions as being functions $y(x)$, despite the fact that such “functions” would not be univalent. Also, a substantial number of students reject constant functions. Finally, an overwhelming majority of the students accept split domain functions. Here, some interesting inconsistencies appear. For example, the function

$$y = \begin{cases} -3 & x < 0 \\ e^x & x \geq 0 \end{cases}$$

is constant on part of its domain, but is still accepted as a function by more than twice as many students as the function $y = 3$. It is also interesting to compare this with another question in the questionnaire, where the students were asked about the possibility of constructing a function which is integer-valued for all non-integers, and non-integer-valued for all integers (this example was found in Vinner & Dreyfus (1989)). About half of the students accept the existence of such a function, and 12 students construct one. But quite a few of those who reject it, do so based on an assumption that a function must be defined by one formula on the whole of its domain, despite having had no problem accepting the piecewise defined function above. On the other hand, of the students who accept this type of function, only two reject the Dirichlet function, so the students who have grasped the idea of arbitrariness appear to have done so in a consistent manner.

The second research question concerns the differences in the understanding of the function concept between engineering and teacher students in the study. The following table (Table 3) shows the distribution of the students’ definitions, divided according to student category.

Table 3. The number of students’ answers in the eight categories, split according to student category (teachers: 14 students; engineers: 20 students)

Category	1	2	3	4	5	6	7	8
Teacher students	1		1		1	3	6	2
Engineering students		1	5	2	7	3		2

We note, that of the teacher students, 8 end up in the two last categories, that is, fail to provide a useful definition. This percentage becomes even larger if the three teacher students aiming for upper secondary school are discounted. They end up one in category 1 and two in category 6. Hence 8 out of 11 prospective compulsory school teachers cannot give a useful definition of the function concept. Of the engineering students, only two fail at this. Conversely, only two teacher students end up in the three operational categories 3, 4 and 5, whereas 14 of the 20 engineering students do so. However, when it comes to applying the function concept, few obvious differences between the

two groups can be seen. The teacher students tend to be less confident about their answers, and there are those among the teacher students who give incorrect answers even to the most straightforward examples. But on certain examples, for instance those concerning univalence, the teacher students perform better than the engineering students.

Instead, the most striking difference is seen in how the students handle the construction of the integer/non-integer function mentioned above. None of the prospective compulsory school teachers are able to give an answer. Indeed, only one student even tries. The rest just answer "*I don't know*", and several claim not to have understood the question. On the other hand, all of the prospective upper secondary school teachers, and most of the engineering students, have given a correct construction, and even those who believe no such function can exist have provided some kind of argument in favour of this view.

Finally, looking at the classification of the students' conceptions of the function concept, obvious differences are seen. Of the teacher students, one has a structural, 4 operational and 9 pre-operational conceptions of function. Among the engineering students one has a function conception which is approaching the structural, 16 have operational and only 3 have pre-operational conceptions of the function concept. If we discount the prospective upper secondary school teachers, the tendency is even clearer. Among the prospective compulsory school teachers, 9 have pre-operational and only 2 have operational conceptions. So it seems fair to say, that the prospective compulsory school teachers in the study have less developed conceptions of the function concept than the engineering students. There also appears to be a difference between different types of teacher students, but the number of prospective upper secondary school teachers participating in the study is too small for me to dare draw any such conclusions. What can be said, however, is that while even the prospective compulsory school teachers with the most developed conceptions of the function concept still have rather inadequate conceptions, the function conceptions of the prospective upper secondary school teachers are among the richest in the study.

DISCUSSION

This study shows that the participating students primarily have operational, and in some cases pre-operational conceptions of function. This agrees well with earlier research on the subject, which has indicated that a reified concept of function is rare among students of mathematics. But, contrary to several earlier studies (e.g. Akkoç & Tall 2005, Meel 2000, Vinner & Dreyfus 1989), the students in this study show no great discrepancies between their definitions and concept images of the function concept. A probable reason for this is the definitions they have encountered during their studies. It

is explicitly stated in (Akkoç & Tall 2005), and implied in (Vinner & Dreyfus 1989), that in Turkey and Israel (where the respective studies were conducted) the structural Bourbaki definition of function is used in schools, something which is not at all the case in Sweden.

But even though the students' concept images tend to agree rather well with their concept definitions, their concept images are not very rich, something which agrees with (Hansson 2006), where it is shown that the function concept is not so well integrated into the general conceptual structure of the students. The lack of more specific concepts, and examples of standard functions, mentioned earlier could be seen as contradicting earlier results regarding the importance of prototypes on the formation of concept images (e.g. Akkoç & Tall 2002; Tall & Bakar 1991), but it could just as well reflect an attempt at generality on the part of the students. Furthermore, several examples of compartmentalization were found. For example almost all students stated that the diagram showing a curve with a loop did not represent a function, but at the same time a majority of the students claimed that $x^2 + y^2 = 4$ represented a function. Here it should also be noted that I make no claim to generality regarding my results. I am well aware that the validity of my study could be greatly enhanced by for example increasing the number of participating students, and also by including interviews with students.

As noted above, several distinct differences between the engineering students and the prospective compulsory school teachers in the study were found, regarding both the function concepts and the answers to certain of the questions in the questionnaire. Before beginning the study, I had some preconceptions about this. Since mathematically interested and gifted students in Sweden tend to study to become engineers rather than teachers, I had expected differences in mathematical ability. But I had expected the teacher students to show greater interest and ability regarding conceptual understanding and expressing mathematical ideas in words, since these are important abilities for the teaching of mathematics. When it turned out that most of the teacher students had taken quite a lot of mathematics at a university level, my hopes were raised further.

But, contrary to these expectations, the conceptual understanding of the prospective compulsory school teachers was less developed than that of the engineering students. Also, their general attitude and low self confidence is cause for concern. A few of them include words like "hard" and "difficult" in their mind maps, and rate their level of certainty below average on all statements. Asked to define the function concept, one student writes: *"Is part of a graph over a coordinate system. Eeeh... I can't explain it."* (T 6) A problematic answer, coming from a teacher-to-be. This uncertainty was most apparent in their answers to the question about the integer/non-integer function. Almost none of the prospective compulsory school teachers even tried to answer this

question. Among the answers were these: “*Firstly, I had to read the question about five times before I understood a little. Then, when I understood a little, I couldn’t picture this function in my head.*” (T 4) and “*I have no idea. I won’t even think about it, since I don’t intend to study functions in any detail.*” (T 13) This last answer I find especially troubling, since it displays an attitude which I have a hard time reconciling with wanting to become a teacher.

One last point I want to make concerns the difference between the prospective upper secondary school and compulsory school teachers. Although the number of prospective upper secondary school teachers participating was very small, and no real conclusions may therefore be drawn about them, I still find the difference between them and the rest of the prospective teachers in the study striking. One thing worth noting is that, at the university where the study was conducted, they take the same classes as mathematics students and engineering students, while the rest of the prospective teachers take classes specially designed for teacher students. One would expect such classes to be more focused on conceptual understanding, for example, but in the light of this study one has to wonder whether these classes are appropriately designed. I find it problematic that future teachers, having taken more than half of the mathematics classes required, have such low mathematical self confidence, such undeveloped conceptual understanding, and such a hard time expressing themselves mathematically.

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