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The Validity of Students' Conceptions of Differentiability and Continuity

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University students' conceptions of differentiability, continuity and relations between the concepts were studied to reveal their choices of representations and their strategies to justify their relational claims. Questionnaires and interviews were used to collect data (questionnaires in the part presented here). The results were analysed and categorized through a framework based on Skemp's (1976) definitions of relational and instrumental understanding, and Tall's (2004) three worlds of mathematics. The students showed ambiguous representations opposing their own statements in some cases. The most common feature among the students to describe a continuous function was incorrect implying a need to develop the students' concept images in that area.

Background

Mathematical proficiency entails ability to justify claims through adaptive reasoning using well selected strategies productively and to understand the concepts involved in the processes (National Research Council, 2001). Students at university level taking their first calculus course deal with many new or already known concepts in a short period of time. They have some pre-knowledge of concepts to consider when learning new aspects of them and they also have to adjust to university studies where the pace is faster and the mathematics deeper than they are used to from upper secondary school. This may have effects on the learning outcome in terms of knowledge depth and tendency to adjust existing representations of concepts. Older insufficient or wrong conceptions may linger in parallel to new and correct ones causing confusion. This was the case in a study about limits of functions where some students thought that limits are unattainable from interpreting the limit definition wrongly at the same time as they stated that a limit was attained in an example (Juter, 2006).

The concepts derivative and continuity are key concepts in calculus and they are closely linked through their definitions. The relation is not symmetrical in terms of implication, i.e. continuous does not imply differentiable but differentiable implies continuous. My aim with this paper is to see how students explain the concepts and the relations between the concepts, i.e. what kind of strategies and representations they use in their explanations, and how they justify their claims by answering the following questions:

1. How do students explain derivatives and continuity?
2. How do students perceive and explain the relations between the concepts?

Theoretical frame

Students' perceptions of mathematical concepts are reflected in their solutions, reasoning and other actions as traces of their concept images (concept image as defined by Tall & Vinner, 1981). Their strategies to justify their mathematical claims have developed in the community of the classroom within its frames of rules and traditions. Understand and being able to solve tasks may be regarded as synonyms for some students, particularly if being able to solve tasks is enough to pass courses. Skemp (1976) distinguished understanding a concept from its core features, relational understanding, which enables implementation of the new concept to existing concept images (which is how Hiebert and Lefevre (1986) defined conceptual knowledge, p 3), from understanding by just being able to perform a particular operation in what he denoted instrumental understanding. Either way to understand a new concept requires mathematical development of existing representations. Tall (2004) introduced a model describing development in three different modes, the conceptual-embodied world with an emphasis on exploring activities, the proceptual-symbolic world focusing concepts' dual features as objects and processes expressed in symbols or procepts (Gray & Tall, 1994), and the formal world where mathematical properties are deduced from the formal language of mathematics in definitions and theorems. Students' concept images develop through the worlds with different emphasis on the three modes allowing them to understand concepts differently. Based on Pinto and Tall's (2001) definitions of formal learner and natural learner together with Tall's three worlds and Skemp's definitions of understanding, a set of categories to classify students' links between concepts was created and used in a prior study, as presented in table 1 (see Juter (2009; 2011) for further details). Examples of classifications of students' links from the earlier study are provided in the table to clarify the categories.

Type of link	**Definition**
Valid link, procedural	True relevant link with focus on calculations or applications, ex: *Derivative of velocity gives acceleration*
Valid link, naturally conceptual	True relevant link revealing a core feature of the concept, not formal, ex: *Derivative is the slope of*

	the tangent in a point
Valid link, formally conceptual	True relevant link formally revealing a core feature of the concept, ex: *If the limit* $\lim_{x \to a} f(x) = f(a)$ *exists in every point then f(x) is continuous*
Irrelevant link, no reason	No actual motivation for the link is provided.
Irrelevant link, no substance	Peripheral true link without substance relevant for the concept, ex: *You can add derivatives*
Invalid link, misconception	Untrue link due to a misconception of the concept, ex: *Continuous means the same change everywhere*
Invalid link, counter perception	Untrue statement contradicting prior statements ex: *sinx is continuous and continuous means linear*

Table 1. Definitions of links between concepts. Examples in italics.

The last four types of links are not desirable for the students, who often are unaware of the quality of the links, particularly if irrelevant or invalid links are mixed with valid ones (Juter, 2011). Links are formed in different situations, e.g. at lectures, with peers or in solitude. Textbooks, lecturers' selections and general interests of the group of students frame the learning environment and therefore affect the representations students are using. Students' abilities, ambitions and confidence also influence their representations. Representations used when learning a certain topic may become vague if they are not endurable enough, e.g. not sturdily linked to other concepts. If a person learns a new mathematical topic in the embodied world and her abilities then develops to symbolic treatment she has changed the way of thinking to a proceptual-symbolic mode (Tall, 2008). If the learning phase in the conceptual-embodied world has been too short or otherwise inadequate, there may become disjoint or vague parts of the concept image rendering the person unable to explain core features of the concept.

Viholainen (2008) conducted a study on students' incoherent conclusions about differentiability and concluded that erroneous conclusions sometimes came from linking correct parts of knowledge in an incorrect manner. He also concluded that erroneous conceptions may come from the individuals' earlier knowledge structures. The student in Viholainen's paper worked with four piecewise defined functions and was asked to determine whether they are differentiable and continuous. The student's first standpoint was that differentiability requires continuity but after thinking about one of the given functions (which is neither differentiable nor continuous) he changed his mind and said that differentiability does not require continuity since he thought the function was differentiable but not continuous. The changed standpoint came from his memory of an invalid method of checking if a piecewise defined function is differentiable. The student did not possess deep enough understanding of the concepts to be able to see what they really mean. Students who recognize the efforts required to make necessary adjustments of their concept images to fully understand a concept may chose to only learn the concept shallowly to be able to manage routine tasks to pass the exam (expressed by a student in a study about students' development of learning limits of functions (Juter, 2006)). All influences on students' mathematical behaviour interact and possibly cause spin off effects.

Methods and sample

In this paper I have chosen to study students' representation choices and proving strategies from openly stated questions. Two groups of students, a total of 43, enrolled in their first calculus course at university level were studied. The first group of students, group A, consisted of 13 males (M) and 9 females (F) (a total of 22 students). The second group, group B, took the same course the following semester and consisted of 16 males and 5 females (a total of 21 students). The duration of the course was 10 weeks and included basic calculus with limits, continuity, derivatives, integrals, differential equations and Taylor's formula. There were a written exam and an oral exam, both individual, to assess the students after the course. The students in both groups were taught by the same lecturer.

The students answered a questionnaire when they had covered continuity and derivatives in the courses. The questions are (the first one is about upper secondary education and omitted here):

2. Explain what a derivative is as if the one you explain to has never heard of the concept.
3. What are derivatives used for?
4. What features do continuous functions have?
5. Does continuity imply differentiability?
6. Does differentiability imply continuity?

The aim was to see how they justified their claims. The questions were posed using only the concepts without examples to allow the students to select whatever way they wanted to explain, hence revealing their perceptions. This strategy is opposed to that used by Viholainen (2008) who started with examples and then asked about general aspects of the properties. A drawback with a general approach is that some students may answer shortly, but the openness of the method enables variety in the descriptions which was the aim. The students' responses to the questions were categorized according to validity and type of arguments used through the categories in table 1. The category *irrelevant link, no* *reason* is not used in this analysis since it is not applicable on this set of data (the students are not asked to explicitly link concepts together. Their links become apparent from descriptions of concepts in a more implicit manner).

After the course, a selection of students from group A (11 of them) were interviewed. The questions were about the questions from the questionnaire and the students' answers, proving, examination forms and attitudes to mathematics. Some of the 11 interviewed students took the following multivariable analysis course and four of them were interviewed again. The focus was the same as before with the experiences of the first course and implications on the second in terms of studying strategies and examination forms. The students from group B were studied at their oral exams to reveal their strategies for proving theorems and reasoning. The studied oral exams were conducted by their lecturer and me. The results from the questionnaire will be presented in this paper.

Results

The results are presented and discussed in the order of the tasks in the questionnaire. The open questions in the questionnaire resulted in a large number of different answers from the students. Some of which occurred in several categories and in some cases one student wrote several explanations so the number of answers in the categories where more than one type of answer is joint sometimes exceed the number of students represented in that category. The students are represented as two groups, group A and B, and the number of answers of each type shows the preferred views of the students in the groups. The most common answers to the questions 2 to 6 are presented with the number of students in each of the groups A and B in brackets after the answers. Examples of students' arguments are provided after questions 5 and 6.

> **Question 2**. Explain what a derivative is as if the one you explain to has never heard of the concept:
>
> **Most common answers:**
> Slope of a curve or tangent, (14 from A, 16 from B of which 3 from A and 3 from B wrote slope of tangent)
> Rate of change or a measure of change, (15 from A, 12 from B)

Two from group A and four from group B described the derivative as a function in itself, describing slope or rate of change of another function. A vast majority of the students explained derivatives generally and used words as change and slope which implies that they have a natural intuitive sense of the concept, *valid link, naturally conceptual* in table 1. Only two students explained derivative with a stronger focus on the processes of calculations or definition. One of them stated that “The derivative is one degree less than the original function”. This student probably refers to polynomial functions and her statement does not show any of the core features of the concept of derivative in general, *is* in table 1. The other student described derivative as “The difference in y divided by the difference in x” which is a core feature of the definition of derivative and hence categorized as *valid link, formally conceptual* in table 1. One student claimed to have “no idea” how to explain what a derivative is. He nevertheless answered the following question about the use of derivatives as a means for dealing with graphs so he had an idea of the use of the concept, but not an explanation he was willing to expose here.

Question 3. What are derivatives used for?

Most common answers
Studies of change or rate of change, study curves, (23 from A, 11 from B)
Maximum and minimum, (5 from A, 11 from B)

Most answers were similar to the answers of question 2, but 6 students used links to other mathematical concepts to describe what derivatives are used for, e.g. limits, integral calculations, inequalities, asymptotes and differential equations (2 students), categorized as *valid link, procedural* in table 1.

Question 4. What features do continuous functions have?

Most common answers
They are differentiable, (8 from A, 8 from B)
They are integrable, (5 from A, 6 from B)

16 students thought that a continuous function is also differentiable in general revealing that there are pieces missing in their concept images of the concepts, *invalid link, misconception* in table 1. 7 students from group A and 3 from group B stated that there are no leaps in the graphs, *valid link, naturally conceptual* in table 1. 7 from group A and 2 from group B stated that the intermediate value theorem is valid and 3 from group A and 4 from B used the definition of continuity showing that the limit in each point is equal to the function value, *valid link, naturally conceptual* and *valid link, formally conceptual* respectively in table 1.

Question 5. Does continuity imply differentiability?

Most common answers
No, (16 from A, 11 from B)
Yes, (8 from A, 5 from B)

The most common way to justify the answer “No” was to give a counter example. 10 students from group A and 4 from B presented $|x|$ as a counter example and one from group A chose e^{x^2} as a counter example. The students’ “Yes” answers had more varied justifications. For example, 4 students (3 from group A and 1 from B) claimed that there is a certain slope of the curve and it is hence differentiable. Left and right limits will be the same was argued by 2 students from group A and 2 from B. There are 24 answers from the 22 students in group A. The reason is that two students answered the question both ways. One wrote: “Yes, if we do not count functions with absolute values”.

If the results from question 4 and 5 are compared it shows that 8 of the students who claimed that a continuous function is differentiable in question 4 also claimed that continuity implies differentiability in question 5. They were non-ambiguous in their conceptions even though they were incorrect. On the other hand, 5 of the students who incorrectly claimed that all continuous functions also are differentiable in question 4 correctly stated that continuity does not imply differentiability in the following question, so their first answers would be categorized *invalid link, counter perception* in table 1. One student simply answerer “No”, another justified his answer by saying that the function need to be harmonic, one stated that differentiability implies continuity, and two had similar reasoning, one about pointy graphs where there is no derivative at the peak and the other stating that there can be different derivatives in the same point.

Question 6. Does differentiability imply continuity?

Most common answers
Yes, (16 from A, 11 from B)
No, (5 from A, 6 from B)

7 students from group B just answered “Yes” without any explanation. None from group A did. One from each group just answered “No”. 6 students from group A correctly used the definitions for derivative and continuity to explain their answers (“Yes”). None of the students from group B used such explanations. Four students (3 from A and 1 from B) used reasoning about connectedness to justify their “Yes”. 5 students thought that a function can be

differentiable but not connected and hence not necessarily continuous (2 from group A and 3 from B).

Discussion

The students' explanations of derivative show a rather uniform view in terms of slope and rate of change, which was an expected result. Derivatives are used for studies of change and rate of change and for studying curves or for determining extreme values for functions according to the students. The former type of answer was given by a majority of the students in group A (23 students) and only a few (5 students) chose the latter type. The students' answers in group B, on the other hand, were evenly distributed in the two categories (11 in each category). This result implies a stronger emphasis on solving problems in group B than in group A. The first category comprises more generally expressed descriptions of what derivatives are used for. Answers to question 6 also reveal a difference between the groups where 7 from group B answered correctly without an explanation and none of the students in group A did. Six students from group A correctly used definitions of derivative and continuity to justify their claims while none of the students in group B did. The students had the same teacher, but the groups showed different characteristics in terms of generality and ability to use theory. The reason for this difference can be that the groups take the course in different semesters so the students are there with different intentions.

The students' explanations of continuity also reveal a trend, but in an undesirable direction as the most common answer was that all continuous functions are differentiable (16 students thought so, 8 from each group). This result was more surprising since they had worked with these properties explicitly in the course. The students had, despite this, not been able to create valid links to help them see the specifics of the concepts.

Justifications used by the students were often not sufficient to be called proofs, mostly due to a lack of arguments. In the cases where proofs were actually correctly conducted (by 7 students in question 4 and 6 students in question 6) almost all proofs were according to the textbook. The answers to questions 4 and 5 show an uncertainty about the concepts and their relation to each other. Five students gave opposing answers to what they had answered before. Two of them used valid arguments, pointing at core features of differentiability, for their claim that continuity does not imply differentiability. They knew that there can only be one derivative at each point of the function for it to be differentiable. This property was not evoked when they answered question 4 about properties of continuous functions. They did think of derivatives when prompted to think about continuity, but they did not think of the crucial aspects of the concepts.

8 of the 16 students claiming that continuity implies differentiability in question 4 kept that standpoint in question 5. The function has a certain slope at all points and is hence differentiable was one argument used by four students in question 5 and two students reasoned about left and right limits being the same. These students used thought experiments to investigate in their minds to be able to answer the question. The examples were, as it seems, chosen without taking properties of differentiability into consideration. A large number of students used the absolute value function as a counterexample to show that continuous functions do not need to be differentiable, 10 from group A and 4 from group B. The example is used as a generic example of a non-differentiable continuous function that helps the students remember the property and understand why there cannot be an unvarying derivative at the peak. There were however an example of a student using the example as an exception to what he thought was the rule that continuity implies differentiability. Either he did not know what properties of the function made it an exception or he knew the properties but not how to use that information in a general context to see that the implication is not true.

Vague memory or misunderstood pieces of knowledge can sometimes be a reason for the students' uncertainty about concepts' properties, as the student in Viholainen's (2008) study exemplifies as well as some of the students in the present study. Different parts of a concept image dominate in different situations which also influences the students' actions which we can see in the ambiguity of the answers of questions 4 and 5. Memory and other cognitive factors fundamentally influence students' learning abilities. An established memory of a concept may be hard to change and even harder to maintain changed. In the formerly mentioned study about limits of functions (Juter, 2006) a couple of students were certain that limits are always unattainable by the function approaching the limit. During a calculus course, this was sorted out and in an interview the students showed that they understood why some limits are attainable and some not. A year after the course was over, the students were interviewed again and their former standpoint was back, i.e. limits of functions are never attainable. The memory of the first view (often linked to their interpretation of the formal definition of limits) dominated over the second view of examples where limits are attainable and a correct interpretation of the definition. It can take a long time to establish new views of already established notions.

This is a first analysis of the data collected. Awareness of the validity of one's own mathematical representations is a first step to improvement of them. Data about the students' habits of study, attitudes to mathematics and assessment has been collected and further analysis is in progress.

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