Draft version, 2012-01-12

Explanation as a Ground for Beauty?

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This work-in-progress report explores the question of whether mathematical explanation is, or should be, linked to the notion of mathematical beauty. If there is a connection between explanation and beauty, it is not a straightforward one - we will provide examples of explanatory proofs which are not beautiful, and beautiful proofs which are not explanatory. Still, the argument goes, there is some essential connection between beauty and explanation, and this connection can help render the question of what mathematical beauty consists in more tractable.

The main distinction we draw upon is between proofs that explain and proofs that demonstrate (Steiner, 1978; Hanna, 1990). Our claim is that proofs that explain are often found more aesthetically pleasing than proofs that merely demonstrate. To illustrate this point we will draw upon several examples from the literature and some pilot data of mathematicians making aesthetic judgements about different proofs. One such example is based on the following question:

> Suppose you decided to write down all whole numbers from 1 to 99999. How many times would you have to write the digit 7?

Consider the following two solutions (adapted from Dreyfus and Eisenberg 1986), the first of which we refer to as a "Bookkeeping" solution, which systematically counts all of the 7's, but does not provide a sense of explanation:

<u>Solution 1</u>: (Bookkeeping)
Between 1-99: 20 (10 in 1's place; 10 in 10's place)
Between 1-999: 300 (20 for each 100's, plus an extra 100 in the interval 700-799)
Between 1-9 999: 4 000 (300 for each 1 000's, plus an extra 1000 in the interval 7000-7999)
Between 1-99 999: 50 000 (4 000 for each 10 000, plus an extra 10 000 in the interval 70 000 – 79999)

The second solution provides more structure than the first, appealing to the symmetric character of the number 7 (it is not privileged over the other digits, we could have very well asked how many 3's there are). This solution is more explanatory than the first, in that it provides a sense of why we get this particular result, and allows for generalization (for instance it is easy to see from this solution how many times the number 7 appears between 1 and 100,000,000).

> Solution 2: Include 0 among the numbers of consideration (this won't change the answer since 7 is not a digit of 0.) Now suppose all numbers from 0 to 99,999 are written down with five digits each, e.g. 306 is written as 00,306. In this set of all combinations every digit will take every position equally often, so every digit must occur the same number of times. There are 100,000 numbers that have 5 digits each, that is 500,000 total number of digits. Each of the 10 digits appears equally often, so each one appears 50,000 times. In particular, this is true for the number 7.

The explanatory proof was rated as more aesthetically pleasing to mathematicians in Dreyfus and Eisenberg's paper. This result was confirmed by the current author in a pilot study conducted with 6 mathematicians, and with a workshop of high school teachers. A similar pilot study was conducted using different proofs of the square root of 2 is irrational, and an explanatory proof was considered more aesthetically pleasing than several non-explanatory proofs[1].

The connection, if there is one, between beauty and explanation is not at all clear. But these pilot studies provide some empirical evidence that a connection exists, perhaps along the lines of what Rota (1997) refers to as "enlightenment." This would add a fourth category to what Natalie Sinclair has listed as three roles of the aesthetic: (1) motivating the choice of certain problems to solve; (2) guiding the mathematician to discovery; and (3) helping a mathematician decide on the significance of the result. The fourth category would be to engender understanding, something we know that explanation provides, while demonstration does not.

References

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[1] For space considerations this example is left out here, but will be included in the presentation.