Nonlinear Problems for Δ_p and Δ

Linköping 2009

Abstracts

The Loewner type estimates for p-modulus of curve families beyond the natural setting $p = n$

Tomasz Adamowicz

We will present various upper and lower estimates for p-modulus of curve families beyond the conformal setting in the Euclidean case $\overline{\mathbb{R}}^n$ for $n-1 < p \leq n$ and apply these results to obtain the p-Loewner type estimates. Next, we will discuss similar results for the setting of Q-Ahlfors regular metric measure spaces supporting $(1, p)$-Poincaré inequality with $1 \leq p \leq Q$ and for $p \geq Q$. Our general discussion encloses the estimates for the modulus of the family of curves passing through the given point, the ring domain modulus as well as the relation between Riesz potentials and p-modulus of curves passing through the point.

The talk is based on the joint work with N. Shanmugalingam.

Existence of three solutions to a class of Neumann doubly eigenvalue elliptic systems driven by a $(p_1, ..., p_n)$-Laplacian

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Abstract

In this work we are interested in multiplicity results for the following Neumann elliptic system

$$\begin{cases} -\Delta_{p_1} u_1 + a_1(x)|u_1|^{p_1-2}u_1 = \lambda F_{u_1}(x, u_1, ..., u_n) + \mu G_{u_1}(x, u_1, ..., u_n) & \text{in } \Omega, \\ -\Delta_{p_2} u_2 + a_2(x)|u_2|^{p_2-2}u_2 = \lambda F_{u_2}(x, u_1, ..., u_n) + \mu G_{u_1}(x, u_1, ..., u_n) & \text{in } \Omega, \\ ... \\ -\Delta_{p_n} u_n + a_n(x)|u_n|^{p_n-2}u_n = \lambda F_{u_n}(x, u_1, ..., u_n) + \mu G_{u_1}(x, u_1, ..., u_n) & \text{in } \Omega, \\ \frac{\partial u_i}{\partial \upsilon} = 0 \text{ for } 1 \leq i \leq n & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $\Delta_{p_i} u_i = \text{div}(|\nabla u_i|^{p_i-2}\nabla u_i)$ is the p_i-Laplacian operator, $p_i > N$ for $1 \leq i \leq n$, $\lambda,\ \mu > 0$, $\Omega \subset R^N (N \geq 1)$ is a non-empty bounded open set with a boundary $\partial\Omega$ of class C^1, $a_i \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega a_i > 0$ for $1 \leq i \leq n$, $F : \Omega \times R^n \to R$ is a function such that $F(., t_1, ..., t_n)$ is continuous in Ω for all $(t_1, ..., t_n) \in R^n$ and $F(x, ., ..., .)$ is C^1 in R^n for almost every $x \in \Omega$, $G : \Omega \times R^n \to R$ is a function such that $G(., t_1, ..., t_n)$ is measurable in Ω for all $(t_1, ..., t_n) \in R^n$ and $G(x, ., ..., .)$ is C^1 in R^n for almost every $x \in \Omega$, F_{u_i} and G_{u_i} denotes the partial derivative of F and G with respect to u_i, respectively, and υ is the outward unit normal to $\partial\Omega$. Our main tool is a recent three critical points theorem of B. Ricceri.

Asymptotic behavior of viscosity solutions for ∞-Laplace parabolic equations

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Abstract

In this talk, the asymptotic behavior of viscosity solutions $u = u(x,t)$ as $t \to \infty$ is discussed for the following Cauchy-Dirichlet problem:

$$u_t = \Delta_\infty u \quad \text{in} \quad Q := \Omega \times (0,\infty), \tag{1}$$

$$u = \varphi \quad \text{on} \quad \partial\Omega \times (0,\infty), \tag{2}$$

$$u = u_0 \quad \text{on} \quad \Omega \times \{0\} \tag{3}$$

with the so-called ∞-Laplacian Δ_∞ given by

$$\Delta_\infty u(x) := \left\langle D^2 u(x) Du(x), Du(x) \right\rangle = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_i}(x) \frac{\partial u}{\partial x_j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x).$$

We deal with the following three cases: (i) $\Omega = \mathbb{R}^N$ and u_0 has a compact support; (ii) Ω is a bounded domain and $\varphi \equiv 0$ (homogeneous Dirichlet case); (iii) Ω is a bounded domain and $\varphi \not\equiv 0$ (inhomogeneous Dirichlet case); notice that we have assumed φ to be independent of the time variable.

This talk is based on a joint work with Petri Juutinen and Ryuji Kajikiya.

Boltyanski's variational technique, Pontryagin's principle and minimax on the line.

Gunnar Aronsson, LiU

The purpose of my talk is to sketch very briefly how Boltyanski's variational technique of "needle variations" (also called "spike variations") under some conditions can be adapted to a minimax problem in one independent variable, and lead to a result similar to Pontryagin's maximum principle. This will also imply some regularity for absolutely minimizing functions. These satisfy a generalised Euler-type equation in a weakened sense.

Optimal Doubling, Reifenberg Flatness and Operators of p-Laplace type

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Abstract

In this paper we consider operators of p-Laplace type, for $1 < p < \infty$, of the form $\nabla \cdot A(x, \nabla u) = 0$. Concerning A we impose, for $p \in (1, \infty)$ fixed, an appropriate ellipticity type condition, Hölder continuity in x and we assume that $A(x, \eta) = |\eta|^{p-1} A(x, \eta/|\eta|)$ whenever $x \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^n \setminus \{0\}$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, let D be a compact subset of Ω. We say that $\hat{u} = \hat{u}_{p,D,\Omega}$ is the A-capacitary function for D in Ω if $\hat{u} \equiv 1$ on D, $\hat{u} \equiv 0$ on $\partial\Omega$ in the sense of $W_0^{1,p}(\Omega)$ and $\nabla \cdot A(x, \nabla \hat{u}) = 0$ in $\Omega \setminus D$ in the weak sense. We extend $\hat{u}$ to $\mathbb{R}^n \setminus \Omega$ by putting $\hat{u} \equiv 0$ on $\mathbb{R}^n \setminus \Omega$. Then there exists a unique finite positive Borel measure $\hat{\mu}$ on $\mathbb{R}^n$, with support in $\partial\Omega$, such that

$$\int \langle A(x, \nabla \hat{u}), \nabla \phi \rangle \, dx = -\int \phi \, d\hat{\mu} \quad \text{whenever} \quad \phi \in C_0^\infty(\mathbb{R}^n \setminus D).$$

In this paper we prove that if Ω is Reifenberg flat with vanishing constant, and if $K \subset \mathbb{R}^n$ is a compact subset, then

$$\lim_{r \to 0} \inf_{w \in \partial\Omega \cap K} \frac{\hat{\mu}(\partial\Omega \cap B(w, \tau r))}{\hat{\mu}(\partial\Omega \cap B(w, r))} = \lim_{r \to 0} \sup_{w \in \partial\Omega \cap K} \frac{\hat{\mu}(\partial\Omega \cap B(w, \tau r))}{\hat{\mu}(\partial\Omega \cap B(w, r))} = \tau^{n-1},$$

for every τ, $0 < \tau < 1$. In particular, we prove that $\hat{\mu}$ is an asymptotically optimal doubling measure on $\partial\Omega$.

2000 *Mathematics Subject Classification.* Primary 35J25, 35J70.

Keywords and phrases: Reifenberg flat domain, Reifenberg flat domain with vanishing constant, p-harmonic function, A-harmonic function, variable coefficients, doubling measure, asymptotically optimal doubling measure.

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Image reconstruction with p-parabolic equations

George Baravdish, Olof Svensson, LiU

In many important engineering problems, for example in the medical area, one has to reconstruct an image from a given blurred version of the image. Due to the importance of this problem many different methods and techniques for deblurring and denoising the image have been proposed. Recently, methods based on PDEs have been suggested.

In this talk, we are going to assume that the blurred image is obtained from a process modelled by the p-parabolic equation with homogeneous Neumann boundary condition. The inverse problem of reconstructing the original image is equivalent to determine the initial condition of the p-parabolic equation given the usual conditions of the direct problem and knowledge of the noisy image given at a later instant of time.

This additional information ensures that the inverse problem has a unique solution, however, the backward p-parabolic equation is an ill-posed model. For the stable image reconstruction we propose and investigate an iterative procedure based on a sequence of well-posed direct problems. Numerical examples are presented showing the feasibility of our procedure applied to e.g. medical images.

From ∞-harmonic to ∞-analytic: Aronsson equations suggested by power-law resistivity

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Abstract

Several models of dielectric breakdown, electrical resistivity, and polycrystal plasticity may be viewed as limiting cases of various power-law models via Γ-convergence; the effective yield set in each case is characterized by means of variational principles associated to supremal functionals acting on fields subject to constant rank differential constraints. I will discuss the derivation of some of these models and indicate the Aronsson equations which play a role in their analysis, including an intriguing system of PDEs arising in the resistivity setting.

Normal forms for surface water waves

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This talk concerns the problem of an incompressible inviscid and irrotational fluid under a gravitational force and in a domain with a dynamic free boundary. This is the classical problem of *ocean* or *water waves*. It has been known since Zakarov (1968) that the problem can be posed as a system of Hamiltonian PDE, and that the equilibrium solution is an elliptic stationary point (in the sense of dynamical systems). We consider the case of two dimensional water waves which are 2π-periodic in the x-variables (and in infinite depth, for the present). Taking the analogy with Hamiltonian mechanics a step further, we develop a normal form for the Hamltonian in a neighborhood of a phase space point $u = (\eta, \xi) = 0$ in the Sobolev space $H = H^r_\eta \times H^{r-1}_\xi$ for $r > 3/2$, which eliminates all quadratic nonlinear terms in the evolution equations. The normal forms transformation is a canonical transformation which mixes the variables $\eta(x)$ describing the fluid domain and those giving the boundary values $\xi(x)$ of the velocity potential. We show that this transformation an analytic symplectomorphism on the space H. This development is the key to several technical advances, which we are pursuing: (i) a long time existence theorem for the water waves equations, (ii) a justification of the NLS modulational scaling regime, and (iii) a new view of the KdV long wave scaling limit.

This is work in progress in collaboration with Borys Alvarez-Samaniego and Catherine Sulem.

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ABSOLUTELY MINIMIZING LIPSCHITZ EXTENSIONS, THE INFINITY-LAPLACE EQUATION, AND ALL THAT

Michael Crandall, Santa Barbara

Below U, V, denote bounded open subsets of $\mathbb{R}^n$, $K \subset \mathbb{R}^n$ is arbitrary and u is a real valued function defined on the sets implicit in the displays. Here are two functionals with "sup" in their definitions:

$$\text{(1)} \qquad \operatorname{Lip}(u,K) := \sup\left\{\frac{|u(x)-u(y)|}{|x-y|} : x,y \in K, x \neq y\right\},$$

$$\text{(2)} \qquad \operatorname{Grad}_\infty(u,V) := \||Du|\|_{L^\infty(V)} := \operatorname{ess\,sup}_{x\in V}|Du(x)|.$$

In (1) and (2), $|\cdot|$ denotes the Euclidean norm. The value ∞ is allowed for both functionals, and it is assigned to $\operatorname{Grad}_\infty(u,V)$ if u is not locally Lipschitz continuous in V. $\operatorname{Lip}(u,V) = \operatorname{Grad}_\infty(u,V)$ if V is convex, but not in general.

Here are associated variational problems: given U and $b \in C(\partial U)$, define the class of "admissible functions"

$$\mathcal{A}_b := \left\{u \in C(\bar U) : u = b \text{ on } \partial U\right\}$$

and consider the problems

$$\text{MinLip}: \qquad u \in \mathcal{A}_b, \quad \operatorname{Lip}(u,\bar U) = \min_{v\in\mathcal{A}_b} \operatorname{Lip}(v,\bar U),$$

and

$$\text{MinG}_\infty: \quad u \in \mathcal{A}_b, \quad \operatorname{Grad}_\infty(u,U) = \min_{v\in\mathcal{A}_b} \operatorname{Grad}_\infty(v,U).$$

Both of these problems have maximal and minimal solutions given by formulas (some of which will be explained), but the maximal and minimal solutions do not in general coincide. There is a lot of nonuniqueness associated with the lack of strict convexity of the functionals involved.

The theory becomes rich when one asks not for minimizers, but for minimizers which are also "absolute minimizers," a notion introduced by Gunnar Aronsson in the 1960's.

The "absolutely minimizing" notion associated to MinLip is

$$\text{AML}: \qquad \operatorname{Lip}(u,V) = \operatorname{Lip}(u,\partial V) \quad \forall\, V \ll U,$$

while that associated to MinG_∞ is

$$\text{AMG}: \quad \operatorname{Grad}_\infty(u,V) \le \operatorname{Grad}_\infty(v,V) \ \ \forall\, V \ll U,\ v \in C(\bar V) \ \ni \ u = v \text{ on } \partial V.$$

Here $V \ll U$ means $\bar V$ is a compact subset of U.

It turns out that the problems MinLip and MinG_∞ have the same unique solution with the associated absolutely minimizing property. If fact, both AML and AMG are equivalent to the property of being *infinity harmonic*, that is

$$\Delta_\infty u := \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i,x_j} = 0,$$

if this pde is understood in the viscosity sense (which will also be explained).

Owing to a recent comparison proof made available by Scott Armstrong and Charles Smart, the comparison theory of AML and AMG functions is now sweet and elementary, after one knows other formulations of the absolutely minimizing properties (in particular the "convexity criterion," which flows from "comparison with cones"). Uniqueness for the Dirichlet problem for the infinity Laplace equation is then a consequence of its easily established equivalence to the absolutely minimizing properties.

In our first two lectures we will attempt to present all this (the equivalences, the comparison result, the existence theorem) with the efficiencies now available via the new comparison proof. We hope that this choice will provide a quite accessible introduction to this very interesting subject for those who are meeting it for the first time, while the organization might interest those who already know the basics.

If time flows as planned (hah!), in our last lecture we will comment on some of the many generalizations, related questions, and connections to other subjects.

The slides from which the lectures will be presented should be available by August 6 at

www.math.ucsb.edu/~crandall/notes/linkslides.pdf

Talk in Linköping - Michela Eleuteri

Title: "p-harmonic functions and obstacle problems: sharp regularity results and generalization to the variable exponent setting"

Abstract

We will present some recent regularity results concerning obstacle problems. We will start with sharp regularity results for local minimizers of functionals with standard growth (including the case of p-harmonic functions); these results are obtained in the setting of Morrey and Campanato spaces. Then we will generalize these results in the variable exponent setting.

Δ_p on the torus: An application to multiscale convergence

John Fabricius

Abstract. The contents of this talk are related to some new results concerning a weak convergence method for nonlinear pde known as *multiscale convergence*, which has its origins in homogenization theory. In particular I will emphasize the importance of the Δ_p-operator in obtaining estimates (in the dual norm) on highly oscillating vector fields belonging to $L^p(\mathbb{T}^N; \mathbb{R}^N)$, where $1 < p < \infty$, that are in some sense "almost divergence free". This is the key ingredient in the proof of one of the most important theorems in multiscale convergence.

Zohra Farnana, LiU

The double obstacle problem on metric spaces

Abstract

During the last decade, potential theory and p-harmonic functions have been developed in the setting of doubling metric measure spaces supporting a p-Poincaré inequality. This theory unifies, and has applications in several areas of analysis, such as weighted Sobolev spaces, calculus on Riemannian manifolds and Carnot groups, subelliptic operators and potential theory on graphs. In this talk we give an overview of the background of Sobolev spaces and p-harmonic functions in metric spaces. In particular we discuss the double obstacle problem. We consider the existence, uniqueness and regularity of solutions of the double obstacle problem. Furthermore we study convergence problems for the solutions.

Continuity of the Saturation in the Flow of Two Immiscible Fluids in a Porous Medium

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Abstract

Two immiscible fluids move in a porous medium, assimilated with a bounded, connected, open set $E \subset \mathbb{R}^N$ with boundary ∂E of class C^1. In order to study their evolution, we have to deal with the weakly coupled system

$$\begin{cases} v_t - \operatorname{div}[A(v)\nabla v + \mathbf{B}(v)] = \mathbf{V} \cdot \nabla C(v) \\ \operatorname{div} \mathbf{V} = 0 \end{cases} \quad \text{in } E \times (0,T), \tag{1}$$

where

$$\begin{aligned} &\mathcal{K}(v) = k_1(v) + k_2(v) && A(v) = \frac{k_1(v)k_2(v)}{\mathcal{K}(v)} p'(v) \\ &\mathbf{B}(v) = \frac{k_1(v)k_2(v)}{\mathcal{K}(v)}[\mathbf{e}_1(v) - \mathbf{e}_2(v)] && C(v) = \frac{k_2(v)}{\mathcal{K}(v)} \quad \text{or} \quad -\frac{k_1(v)}{\mathcal{K}(v)} \\ &\mathbf{V} = \mathcal{K}(v)[\nabla u + \mathbf{e}(v)] && \mathbf{e}(v) = \frac{k_1(v)\mathbf{e}_1(v) + k_2(v)\mathbf{e}_2(v)}{\mathcal{K}(v)}, \end{aligned}$$

and $k_i(v)$ are the permeabilities of the two fluids, $\mathbf{e}_i(v_i)$ their gravity forces, $p(v)$ the capillary pressure, subject to the condition

$$p_{min} \le p(v) \le p_{max} \tag{2}$$

for given constants

$$-\infty \le p_{\min} < 0 < p_{\max} \le +\infty. \tag{2$'$}$$

The system consists of an elliptic equation and a degenerate parabolic equation. The unknown functions u and v and the equations they satisfy, represent the pressure and the saturation respectively, subject to Darcy's law and the Buckely-Leverett coupling. Due to the empirical nature of these laws no determination is possible on the structure of the degeneracy exhibited by the system. It is established that the saturation is a locally continuous function in its space-time domain of definition, irrespective of the nature of the degeneracy of the principal part of the system.

This is a joint work with Emmanuele DiBenedetto and Vincenzo Vespri.

EXISTENCE AND STABILITY OF FULLY LOCALISED THREE-DIMENSIONAL GRAVITY-CAPILLARY SOLITARY WAVES

Mark Groves, Saarbrucken

A solitary wave of the type advertised in the title is a critical point of the Hamiltonian, which is given in dimensionless coordinates by

$$H(\eta,\xi)=\int_{\mathbb{R}^2}\left\{\frac{1}{2}\xi G(\eta)\xi+\frac{1}{2}\eta^2+\beta\sqrt{1+\eta_x^2+\eta_z^2}-\beta\right\},$$

subject to the constraint that the impulse

$$I(\eta,\xi)=\int_{\mathbb{R}^2}\eta_x\xi$$

is fixed. Here $\eta(x,z)$ is the free-surface elevation, ξ is the trace of the velocity potential on the free surface, $G(\eta)$ is a Dirichlet-Neumann operator and $\beta>1/3$ is the Bond number.

In this talk I show that there exists a minimiser of H subject to the constraint $I=2\mu$, where $0<\mu\ll 1$. The existence of a solitary wave is thus assured, and since H and I are both conserved quantities its stability follows by a standard argument. 'Stability' must however be understood in a qualified sense due to the lack of a global well-posedness theory for three-dimensional water waves.

Curvature flow connections with the Δ_∞

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Abstract We will begin by examining the connections between solutions of the Δ_∞ in n dimensions and certain types of inverse sectional curvature flow problems. In particular we will see how solutions of the Δ_∞ can be constructed from smooth solutions of these inverse sectional curvature flows. In the second part of my talk we will examine the properties that this connection imposes on the viscosity solutions of the Δ_∞ in 2 dimensions.

Game solutions of non-linear partial differential equations

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Abstract The Δ_∞ was derived in some sense as the Euler-Lagrange equation for the absolute minimizer of the norm of the gradient, $|Du|$. Similarly the so-called Aronsson equation was derived for the absolute minimizers of the more general function of the gradient, F(x,u,Du). I will describe how these PDE operators and their generalizations arise in the context of optimal control and two-person game theory. This represents a new characterization of such problems since a number of the generalizations simply cannot be obtained as Aronsson equations for L^∞ variational problems.

Nonexistence results for A-harmonic problems

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Abstract

Our purpose is to discuss how to obtain nonexistence results for solutions of differential inequalities

$$-\Delta_A u \geq \Phi(u), \tag{1}$$

where u is supposed to be nonnegative and defined on $\mathbf{R}^n$, $\Delta_A u = \operatorname{div}(A(\nabla u))$, $A(\lambda) = B(|\lambda|)\lambda : \mathbf{R}^n \to \mathbf{R}^n$, Φ is strictly increasing and $\Phi(0) = 0$. We assume additionally that the involved function A is such that the function $\bar{A}(\lambda) = B(|\lambda|)|\lambda|^2$ is a N-function and satisfies the $\Delta^{'}$-condition: $\bar{A}(\lambda_1\lambda_2) \leq D_A\bar{A}(\lambda_1)\bar{A}(\lambda_2)$. When $\bar{A}(\lambda) \geq C\lambda^p$, $p > 1$, and $\Phi(\lambda) = \lambda^q$, the result was already studied in the paper by Pohozhaev and Mitidieri [2]. Our approach is an extension of this classical result to Orlicz setting. We illustrate our approach within functions $\bar{A}$ and Φ of power-logarithmic-type. We also obtain new appriori estimates for solutions of (1). The result is based on joint work [1].

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Superparabolic functions

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Abstract: We discuss potential theoretic aspects of degenerate parabolic partial differential equations of p-Laplacian type. Solutions form a similar basis for a nonlinear parabolic potential theory as the solutions of the heat equation do in the classical theory. In the parabolic potential theory, the so-called superparabolic functions are essential. For the ordinary heat equation we have supertemperatures. They are defined as lower semicontinuous functions obeying the comparison principle. The superparabolic functions are of actual interest also because they are viscosity supersolutions of the equation. We discuss their existence and regularity properties.

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The Benjamin–Lighthill conjecture for near-critical values of Bernoulli's constant

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Benjamin and Lighthill [1] made a conjecture concerning the classical nonlinear problem of steady gravity waves on water of finite depth. This conjecture says that a point of some cusped region on the (r, s)-plane (see Fig. 2 in [1]) corresponds to every steady wave motion described by the problem; here r and s are the non-dimensional Bernoulli constant and the flow force, respectively. On the contrary, at least one steady flow corresponds to every point of the region. In the present talk, we outline how to prove this conjecture for near-critical flows (when r is close to one).

Another question to be considered concerns the uniqueness of solutions. In particular, only the following waves do exist for every near-critical value of r: (i) a unique (up to translations) solitary wave; (ii) a family of Stokes waves (unique up to translations) parametrised by the distance from the bottom to the wave crest. The latter parameter belongs to the interval bounded below by the depth of the subcritical uniform stream and above by the distance from the bottom to the crest of solitary wave corresponding to this r.

Proofs of the above results essentially rely on the integro-differential equation proposed in [2] and fundamental bounds for steady water waves obtained in [3]. The talk is based on joint results with V. Kozlov (Linköping) to be published in [4].

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Boundary Integral Operators and Boundary Value Problems for Laplace's Equation

by John Lewis

Abstract: In this talk I will discuss joint work with TongKeun Chang concerning boundary single and double layer potentials for Laplace's equation in certain bounded d Ahlfors regular domains, considerably more general than Lipschitz domains. We show that these layer potentials are invertible as mappings between certain Besov spaces and thus obtain layer potential solutions to the Regularity, Neumman, and Dirichlet problems with boundary data in these spaces.

The two-phase membrane problem with coefficients below the Lipschitz threshold

Erik Lindgren, KTH

In this talk I will discuss regularity properties of the two-phase membrane problem. I will try to explain how the two-phase problem differs from the one-phase problem and what extra difficulties this leads to. A technique that can be used to attack this problem when the coefficients are not Lipschitz will also be presented.

A Curious Equation Involving the Infinity-Laplacian

Peter Lindqvist

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The problem

$$\min_u \max_x \left(|\nabla u(x)|^{p(x)} \right)$$

leads to a curious partial differential equation involving the celebrated Infinity-Laplacian operator. The solutions are "constructed" through a variational procedure and their uniqueness is derived from the theory of viscosity solutions. Among other devices an interesting approximation of the identity is used. –This is joint work with T. Lukkari.

WOLFF POTENTIAL ESTIMATES FOR ELLIPTIC EQUATIONS WITH NONSTANDARD GROWTH

TEEMU LUKKARI

We discuss potential estimates for elliptic equations with nonstandard growth conditions involving a variable exponent $p(x)$. The prototype of such equations is the $p(x)$-Laplacian

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = 0.$$

More spesifically, superharmonic functions related to a partial differential equation can be characterized as solutions of a nonhomogeneous equation involving a positive measure μ on the right hand side, and their pointwise behaviour can be controlled by means of an appropriate nonlinear potential of the measure μ. For equations similar to the p-Laplacian, these facts were established by Kilpeläinen and Malý. In our case, the appropriate potential turns out to be the variable exponent Wolff potential

$$\mathcal{W}^{\mu}_{p(\cdot)}(x,R) = \int_0^R \left(\frac{\mu(B(x,r))}{r^{n-p(x)}}\right)^{1/(p(x)-1)} \frac{\mathrm{d}r}{r},$$

which is the constant exponent Wolff potential taken pointwise.

This is joint work with N. Marola and F-Y. Maeda.

The Boundary Harnack Inequality for Solutions to Equations of Aronsson type in the Plane

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July 23, 2009

Abstract

In this paper we prove the boundary Harnack inequality for positive functions which vanish continuously on a portion of the boundary of a bounded domain $\Omega \subset \mathbf{R}^2$ and which are solutions to a general equation of p-Laplace type, $1 < p < \infty$. We also establish the same type of result for solutions to the Aronsson type equation $\nabla(F(x, \nabla u)) \cdot F_\eta(x, \nabla u) = 0$. Concerning Ω we only assume that $\partial\Omega$ is a quasicircle. In particular, our results generalize the boundary Harnack inequalities in [BL] and [LN] to operators with variable coefficients.

2000 *Mathematics Subject Classification.* Primary 35J25, 35J70.

Keywords and phrases: boundary Harnack inequality, p-Laplace, A-harmonic function, infinity harmonic function, Aronsson type equation, quasicircle.

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Quasiminimizers – definitions, constructions, and capacity estimates

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Let Ω be an open set in $\mathbf{R^n}$ and $K \geq 1$, $p > 1$. Denote by $W^{1,p}_{loc}(\Omega)$ the first order Sobolev space whose functions and their distributional partial derivatives are locally L^p–integrable in Ω. A K–quasiminimizer u is a function of the class $W^{1,p}_{loc}(\Omega)$ which almost minimizes the p–Dirichlet integral in the sense that

$$\int_{\Omega'} |\nabla u|^p \, dx \leq K \int_{\Omega'} |\nabla v|^p \, dx$$

for all open sets $\Omega' \subset\subset \Omega$ and for all functions v such that $v - u \in W^{1,p}_0(\Omega')$, i.e. the function v has the same boundary values as u in Ω'.

Quasiminimizers cover a wide range of applications and their properties are based only on the minimization of the variational integrals instead of the corresponding Euler equation. For example, regularity properties as Hölder continuity and L^p–estimates are consequences of the quasiminimizing property. The theory of quasiminimizers can be easily extended to metric measure spaces because only the absolute value of the gradient is used and not more subtle properties of the derivative. The theory of quasiminimizers also shows which properties of p–harmonic functions and other potential functions are stable under perturbations of energy changes.

In these lectures we consider the following aspects of quasiminimizers in $\mathbf{R}^n$, $n \geq 1$:

- Definitions
- Quasiminimizers in $\mathbf{R}$
- Constructions for quasiminimizers
- Capacity estimates
- Open problems

For capacity estimates the theory of one dimensional quasiminimizers is used.

Variational methods in the theory of perfectly plastic fluids

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Abstract

The stationary motion of an incompressible fluid is governed by the system of PDEs

$$\nabla \cdot \mathbf{u} = 0, \qquad -\nabla \cdot S + \nabla p = \mathbf{f} \tag{1}$$

where $\mathbf{u} = (u_1, \ldots, u_n)$ velocity, $S = \{S_{ij}\}$ deviatoric stress, p pressure, $\mathbf{f}$ external force. We consider the following constitutive law (R. von Mises (1913)):

$$D = 0 \Longrightarrow |S| \leq g, \quad D \neq 0 \Longrightarrow S = \frac{g}{|D|} D \tag{2}$$

($D = D(\mathbf{u}) = \{D_{ij}(\mathbf{u})\}$, $D_{ij}(\mathbf{u}) = \frac{1}{2}\,(\partial_j u_i + \partial_i u_j)$ rate of strain, $g = \text{const} > 0$ yield value). The relations (2) model perfect plastic behavior of an incompressible fluid („von Mises solid").

The weak formulation of (1), (2) in a bounded domain $\Omega \subset \mathbb{R}^n$ under Dirichlet boundary conditions on $\mathbf{u}$ leads to the problem

$$\text{minimize} \qquad \mathcal{F}(\mathbf{u}) :=\ g \int_\Omega |D(\mathbf{u})| - \int_\Omega \mathbf{f} \cdot \mathbf{u}.$$

We solve this problem in the space $\mathrm{BD}(\Omega)$ under a physically motivated smallness assumption on $\mathbf{f}$. Our method of proof consists in approximating (2) by the power law

$$S_\varepsilon = g|D|^{\varepsilon - 1} D \qquad (\varepsilon > 0)$$

and carrying out the passage to the limit $\varepsilon \to 0$.

Regularity and Free Boundary Regularity for the p-Laplace Operator in Reifenberg Flat and Ahlfors Regular Domains

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June 18, 2009

Abstract

A classical result concerning the harmonic measure ω, due to Lavrentiev, states that if $\Omega \subset \mathbf{R}^2$ is a chord arc domain, then ω is mutually absolutely continuous with respect to the surface measure σ, i.e., $d\omega = kd\sigma$ where k is the associated Poisson kernel. Moreover, Lavrentiev proved that $\log k$ is in the space of functions of bounded mean oscillation, defined with respect to σ, on $\partial\Omega$. Furthermore, later Pommerenke proved that $\Omega \subset \mathbf{R}^2$ is vanishing chord arc if and only if $\log k$ is in the space of functions of vanishing mean oscillation, defined with respect to σ, on $\partial\Omega$. Concerning higher dimensional analogues of the results of Lavrentiev and Pommerenke, such results have been established by Carlos Kenig and Tatiana Toro in a sequence of papers. Furthermore, recently John Lewis and I have established appropriate versions, valid for the p-Laplace equation, $1 < p < \infty$, of the results proved by Kenig and Toro. While the results of Kenig and Toro concern harmonic functions and harmonic measure, i.e., the case $p = 2$, our results are valid for the whole range $1 < p < \infty$ and our results are completely new in the case $p \neq 2$, $1 < p < \infty$. Consequently we have also establish versions, valid in all dimensions, for the p-Laplace equation, $1 < p < \infty$, of the classical results of Lavrentiev and Pommerenke mentioned above. The purpose of the talk is to discuss these recent results.

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Transient thermal radiative convection flow of a heat transfer past a continuously moving porous boundary

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Abstract

An analysis is carried out to study the effect of radiation on the flow and heat transfer past a continuously moving porous plate in a stationary fluid. The solution of boundary value problem has been obtained analytically. The effects of various parameters like magnetic parameter, suction or injection parameter, radiation parameter and Prandtl number on the velocity and temperature profiles as well as the skin-friction coefficient and wall heat transfer are presented graphically and in tabulated form.

The asymptotic behavior as $t \to \infty$ of the components of solution of the Cauchy problem describing small fluctuations of stratified fluid rotation in the semi-space.

Elena Sviridova

Voronezh State University

Abstract. This work is devoted to studying a number of components of solution of the system of partial differential equations. The system of equations describes the small fluctuations of the exponentially stratified and uniformly rotating fluid in the Cartesian system of coordinates (x_1, x_2, x_3) rigidly connected with the rotating fluid. The fluid is stratified along the axis Ox_3 coinciding with the axis of rotation: $\rho_0(x_3) = A\exp(-2\beta x_3)$, where $\beta > 0$ is a parameter of stratification. The existence of solution of Cauchy problem for this system is proved. The estimates of the components of solution and their derivatives in Sobolev spaces are obtained. The asymptotics of the components of the solution are constructed.

Key words: differential equations, systems of differential equations, variable coefficients, existence of solution, asymptotic behavior.

Bernoulli Free-boundaries

J. F. Toland

Linköping August 2009

Abstract

When a domain in the plane is specified by the requirement that there exists a harmonic function which is zero on its boundary and additionally satisfies a prescribed Neumann condition there, the boundary is called a Bernoulli free boundary. The boundary is "free" because the domain is not known *a priori* and the name Bernoulli was originally associated with such problems in hydrodynamics. The classical Stokes wave problem in hydrodynamics is one of the most famous example of problems of this kind. Questions of existence, multiplicity or uniqueness, and regularity of free boundaries for prescribed data need to be addressed.

In these lectures an equivalence will be established between Bernoulli free-boundary problems and a class of equations for real-valued functions of one real variable. We imposes no restriction on the amplitudes or shapes of free boundaries, nor on their smoothness. Therefore the equivalence is global, and valid even for very weak solutions.

An essential observation here is that the equivalent equations can be written as nonlinear Riemann-Hilbert problems and the theory of complex Hardy spaces in the unit disc has a central role. An additional useful fact is that they have gradient structure, their solutions being critical points of a natural Lagrangian. This means that a canonical Morse index can be assigned to free boundaries and the the Calculus of Variations becomes available as a tool for the study.

Some rather natural conjectures about the regularity of free boundaries remain unresolved.

The lectures are closely based on

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On Wolff's anti-Fatou theorem for p-harmonic functions

Harri Varpanen

In 1984, Tom Wolff constructed a bounded p-harmonic function in the upper half-plane having radial limits almost nowhere on the boundary of the half-plane. He also conjectured that the construction generalizes to other domains (such as the disk). I will describe Wolff's construction and talk about the ongoing work of verifying the conjecture for the disk.

Regularity of p-harmonic functions in the Heisenberg group

Xiao Zhong, Helsinki

I will talk about the regularity of p-harmonic functions in the Heisenberg group. Open problems in this topic will be mentioned.