Mathematatish –
a Tacit Knowledge of Mathematics

Håkan Lennerstad, Blekinge tekniska högskola
Lars Mouwitz, Göteborgs universitet

Purpose and method

The purpose of this paper is to highlight the symbolic notations of mathematics and to present some hypotheses. We will stress the language aspect of the symbolic notation system, and therefore we call it "Mathematish". This definition is of course rather vague, but we try to specify our use of the term in Section 2.

There is already a lot of research done about mathematics representations in mathematics education, also using linguistic tools. A short overview is presented in Brown (2001). Also in Nordic mathematics education some research is done in the semiotic field (e.g. Bergsten, 1999; Engström, 2002) and (Winsløv, submitted). Despite this our hope is that our approach to consider mathematics symbolism as a fully developed *language* could create some fruitful hypotheses for this field.

Our method is a rather speculative reasoning, but with some empirical underpinnings from the history of mathematics, from our teacher experience, and from findings in mathematics education research. Some of our arguments are supported by references to the philosophical and linguistic fields of knowledge.

In Section 1 the historical evolution of Mathematish is described briefly. Section 2 is devoted to the various properties of Mathematish as a language, by reference to semiotics and linguistics and by comparisons to natural languages. Section 3 describes aspects of mathematics content and ways that content and Mathematish interact in learning situations.

Background and need for articulation

Mathematish is a very young language compared to other languages. Many researchers have contributed to Mathematish in the form of different shorthand notations replacing expressions of natural languages. Thus Mathematish initially inherited some structure from natural languages. Mathematicians and philosophers like Leibniz and Descartes have promoted the idea of a full-blown formal language, and Mathematish has henceforth grown by further linguistic innovations by researchers fuelled by its sheer efficiency. The success for Mathematish in research and technical applications is overwhelming. This has reshaped mathematics into a subject of formal calculation. Interpretations, typical of rhetoric mathematics are often omitted, which moves "content" to the background.

David Hilbert and the formalists attempted to formalize all mathematics – for them mathematics *is* formal calculation. One central ambition of the formalists was to axiomatize mathematics, i.e. to investigate the formal foundations in order to make calculations as reliable as possible (Davis & Hersh, 1980). A special feature of Mathematish is therefore that its grammar is designed to make it possible to deduce true statements without involving content during the deduction process (see Kline, 1980; Davis & Hersh, 1980).

In 1931 Kurt Gödel demonstrated a limit of axiomatization for mathematics in his incompleteness theorem. A conclusion is that relevant parts of mathematics cannot be formalized. Nevertheless, as symbolic mathematics evolved, the dominance of formalized mathematics increased. The role of relevance or meaning of mathematical concepts and goals of mathematics research in research papers decreased, as well as explicit texts about how to construct proofs in Mathematish. In textbooks proofs often were ready-made, and intuitive and strategic aspects not expressed. It may be considered as a main part of mathematical content – strategies, ideas and methods of how to use the rules. Note that such questions cannot be answered or described in Mathematish; the symbolic notations are not constructed to have themselves as references. Mathematical content of this kind occurs today mostly verbally among researchers and experienced teachers.

The success of Mathematish has inevitably reshaped school mathematics. Industry has posed a need for engineers who can read and handle the mathematical formalism, perhaps underestimating the notion that a successful use of mathematics requires also reflection about content. The needs of mathematics researchers have formed the dominant description of mathematics today.

In this paper we focus on the Mathematish issue to develop alternative descriptions. Initially we unfold the idea that Mathematish *is* a language of its own, and thereby we obtain the possibility to use analogies with natural languages and tools from semiotics and the philosophy of language.

Purposes of Mathematish articulation

By Mathematish articulation we mean descriptions where the immediate purpose is not to provide understanding of mathematical concepts, but on mathematical symbols and their use. This includes general and specific rules and habits in the grammar of Mathematish, and how ways of writing in Mathematish correspond to mathematical ideas.

Firstly, if we regard mathematics as a natural human capability which can be expressed in many individual ways, while Mathematish is the official language, it is very natural that children when starting school do not meet mathematics for the first time in their mathematics class – they merely meet the Mathematish formulation of mathematics for the first time. Furthermore, it is well known that many people in practical occupations with no higher education are able to solve mathematical problems that occur in their occupation. Perhaps these solutions may not be regarded as mathematics since Mathematish is not used (Löthman, 1992). Sometimes the incultivation in Mathematish seems to *decrease* adult students´ capabilities to solve problems (Alexandersson, 1985). Mathematish articulation may help people to recognize their mathematical capabilities. This requires mutual translations between academic mathematics culture and informal mathematical knowledge embedded in practice.

Secondly, the lack of articulation could be a reflection of Mathematish as being an unknown foreign language for many students. Of course there is a constant struggling to learn Mathematish in the mathematics classroom, but the shortcomings may depend on that the teachers themselves have not thoroughly recognized its grammar, especially not compared its grammar to grammars in natural languages. This may depend on that mathematics teachers not so often are trained in linguistic methods. Here we can also see a mother tongue teaching paradox: the more fluently the teacher speaks the language, the more invisible (and unnecessary!) the grammar structure tends to be for him.

Thirdly, in mathematics research there is an intricate interplay between Mathematish formulation and development of both proof ideas and new concepts. Mathematish articulation may clarify this interplay. The history of mathematics gives us many examples of how Mathematish can enforce an introduction of a new concept. The mathematicians will rather stay to the symbolic manipulation rules, the grammar, and gradually accept for instance negative and imaginary number than alter these rules (Kline, 1980). Of course it is sometimes also the other way around; a new concept enforces an introduction of a new symbol and how to handle it. A historical example is the introduction of symbols in differential calculus made by Newton and Leibniz. This dynamic interplay between grammar and thought gives interesting perspectives on the history of mathematics and also on learning situations.

Fourthly, articulation of Mathematish may play a role when discussing the nature of mathematics in general. This is important in its own because of the central position of mathematics in science and society. Other properties of mathematics may become visible by means of the Mathematish-content point of view. One important aspect is the interplay between language and culture, analyzed by structuralists and poststructuralists (de Saussure, 1916; Derrida, 1976).

Properties of Mathematish

Mathematical texts are bilingual

The perspective and focus of this paper is mathematics as a subject having two sides that relate in complicated ways: its general and abstract concepts, and its special symbolic language. This is reflected in the fact that mathematical texts are *bilingual*. By this we mean that some parts are written in natural language, extended with a mathematical terminology, and some parts are written in the mathematical symbolic language, typical for arithmetic, algebra, and analysis. As a simple example we choose the following text line:

A linear equation is one that can be written $ax + by + c = 0$.

The first part is following ordinary language grammar and the very signs used are symbols for *phonemes*, representing the spoken language. The whole structure of the signs used is therefore phoneme based. Words, concept representing clusters of signs in ordinary language, are organized along ordinary language grammar, and the word structure is following ordinary spoken language.

The second part of the line is using other signs, not representing phonemes but mathematical concepts. The "grammar", the rules for ordering these signs, is very different from the grammar of an ordinary language. The structure of the symbols used is following rules of mathematics, a "grammar", especially constructed for this purpose.

Knowledge of Mathematish is knowledge of its grammar: to recognize correct formulas and correct rules to change formulas. Mathematish knowledge is knowledge in "pure formalistic manipulation". We consider neither purposes, goals nor meanings of the manipulations as parts of Mathematish knowledge – this is content.

The idea that mathematics texts are bilingual is not new, for instance is this idea very important in Wittgenstein's philosophy of mathematics, here cited in Waismann (1979):

> ... what is caused to disappear by (a critique of foundations) are names and allusions that occur in the calculus, hence what I wish to call *prose*. It is very important to distinguish as strictly as possible between the calculus and this kind of prose. (p. 149)

With the term "calculus" Wittgenstein included both arithmetic and algebra. We will not follow Wittgenstein all the way to his rather extreme position that "calculus" (*Kalkul) is* the real mathematics, and that "prose" (*Prosa)* is merely confusing and blurring (Marion,1998), but we find his distinction fruitful. Narrative natural language (rhetoric) as occurring in a mathematics text, extended with mathematical terminology, we will call *mathematical prose*.

Of course mathematics also has other types of representations, for instance pictures, graphs and schemas, but in this paper we focus on the two languages mentioned above, and especially on Mathematish. One reason for this focus is that both *mathematical prose* and Mathematish are established vehicles crucial for problem solving and proof activities in both school mathematics and mathematics research, and both have a language character.

The main purpose and aim of this paper is to discuss the following question: Is it fruitful for mathematics education and mathematics research to study Mathematish with similar linguistic tools that are used to study natural languages; could analogies with natural languages create interesting hypotheses?

Mathematics terminology versus Mathematish

Physics, literature, mathematics - most sciences have a terminology of its own. Specialized texts in these subjects may be unreadable for laymen. The specialized terminology is an extension of the vocabulary and is used within the grammar of the natural language. Such a specialized text may be unreadable by laymen also because of unknown figures of thoughts or unknown references. However, if the grammar is different, a person needs to learn not only new words and their meanings, but also new rules of the language. There are certainly many other specialized languages than Mathematish, such as musical notation, molecular notation in chemistry and the Labanotation in dance.

There is a mathematics terminology with is not a part of Mathematish: words such as “addition”, “real number”, “continuous”, “differential equation”, etc. But texts with no formulas, i.e. with no Mathematish, are normally not considered as mathematics texts. Conversely, mathematics texts do not consist of Mathematish only, and no natural language. They are bilingual, and switches from one language to the other are frequent and often unannounced. Some statements can be made in any of the two: Mathematish or English. Some mathematics authors use this possibility to explain Mathematish. However, in a mathematics text the two languages are mostly used for different purposes. While Mathematish is used to specify and manipulate quantitative relations, English is mainly used to describe the logic in the argument, as well as purposes, connections to other results, analogies, images, examples and applications.

Comparisons of Mathematish and ***English***

The sentence “1 + 1 = 2” is a true statement, “1 + 1 = 3” is a false statement, and “1 + 1 = +%” is no statement at all, it is meaningless. The first two follows the grammar of Mathematish, and are either true or false. The third does not follow the grammar. Then it is not a statement and can not be assigned a truth value. It is only a sequence of signs. Note that this grammar is tacit: it is not easy to say which rule is violated. A rule need to be constructed, such as: “on both sides of an equal sign there has to be symbols for numbers or variables”.

This is similar to the sentences “A frog has four legs”, “A frog has seven legs” and “A frog legs”. The first two follow the grammar of English, and we can (in principle) decide if it is true or false. The third “sentence” is meaningless.

It is important to observe that the truth of “A frog has four legs” or “A frog has seven legs” cannot be decided within the language itself. In this case one must import knowledge of biology to decide the truth/falsity. A natural language does not contain truths which they describe, with the exception of analytic truth (rhetoric logic). Note that this grammar is not tacit. “A frog legs” is no sentence since it has no verb. Someone who has English as mother tongue can probably formulate such a rule, even no explicit rule is needed to say that “A frog legs” is no sentence.

The grammar of a natural language does not follow the very structure of the empirical world, and indeed our views of that structure are changing over time. Therefore you cannot deduce new truths (except analytical) about the empirical world with natural language. Mathematish, on the other hand, has a specially constructed grammar following the structure of mathematics, which is mostly numerical and logical. If you start with true premises it is possible to deduce true mathematical sentences within Mathematish without “checking” with mathematics on an outside concept or idea level. This is for instance crucial when you are trying to prove a conjecture. In an ordinary proof, conceptions, intuition and metaphors are (afterwards) “cleaned out” and replaced by Mathematish. An important attendant question is therefore to what degree Mathematish in fact is constituting the mathematical world of concepts and theories. Could it also be fruitful to analyze the claim for using Mathematish when proving as an act of power from the established mathematical discourse, in the meaning of Foucault? See (Foucault,1961).

You could talk about “good Mathematish” in the same way as “good English”. Both good and bad Mathematish are following the grammatical rules, but good Mathematish presupposes a “cultural” knowledge and a feeling for the context. An example is that it is “better” to write $ax + by + c = 0$ instead of $xa + yb + z = 0$ for representing a line. Another example is to know that the parentheses in $f(x + h)$ and in $a(b + c)$ probably have different roles, even if you do not know the actual contexts embedding the expressions. How much do teachers take it for granted that students master not only Mathematish but also “good Mathematish” in the classroom?

You can even identify “dialects” in Mathematish; small differences in how to use symbols and following rules. This is very apparent when comparing textbooks from different countries. Is there a learning problematic with dialectal Mathematish in translated books, or for students not sharing the teachers dialect?

Mathematish – a typical language?

The theoretical background underpinning our question on the role of Mathematish in mathematics, is that the strictly regulated system of arithmetic and formula handling that has emerged in mathematics in many ways *has* the features of a language: it is using a special set of *signs*, the use of the signs is regulated by a *grammar (syntax)*, and it is possible to *produce, interpret* and *translate* propositions designed with these signs and grammar. *Mathematish* also has one of the most powerful “design features” typical for a language; the *double articulation* (or *duality of* patterning), see (Hjelmslev, 1961). This double articulation enables a semiotic code to form an infinite number of meaningful combinations using a small number of low-level units, which in themselves are meaningless. For instance is the use of x, y and z as signs for variables a mere convention started by Descartes when he chose the letters at the end of the alphabet for variable signification.

All these language features open for using methods and perspectives from well elaborated discourses in linguistics, semiotics and the philosophy of language. Our conjecture is therefore that it is fruitful to identify *Mathematish* as mentioned above, not only as representations but as a language of its own.

Tacitness of Mathematish

There is a risk that teachers, well incultivated in Mathematish, will focus merely on content presupposing that the students already master the language. As a result, the structure and the rules of Mathematish will remain largely tacit.

We use the concept "tacit" with the same meaning as in Polanyi (1967), that the knowledge is *not formulated but perhaps possible to formulate*. Some tacit knowledge is possible and also relevant to formulate by language, but other parts are better to *show in practice*. We can also imagine that there could be a kind of knowledge that is neither possible to formulate nor to show, but it is not clear if this should be called knowledge. For an elaborated analysis of tacit knowledge, see (Molander, 1996). Molander identifies a third kind of tacit knowledge, a knowledge that is suppressed to silence. A rather common experience in adults education is that adults' informal knowledge is suppressed by for instance a teacher's claim for Mathematish representation (Nunes et al, 1993).

Another relevant distinction named already in Ryle (1949) is *knowledge-how* and *knowledge-that*, both could be tacit but the latter more easy to formulate: even if you know the rules of a game (knowledge-that) it is not sure that you are an expert in playing the game (knowledge-how), and it could be hard to express this expert knowledge in words. A strong remark is made by Wittgenstein (1983), that there cannot be a rule that also includes how to use the rule.

Mathematish and mother tongues

A good knowledge of formula manipulations can be compared to knowledge of a mother tongue; it is used without any explicit translating processes. It is well known that the structure of a mother tongue is naturally tacit for the user. It is "tacit" in the meaning that it is not expressed or reflected upon, and perhaps some parts are not even expressible. If mathematics teachers use Mathematish similar to a mother tongue, they may mistakenly see the translation problem merely as a concept understanding problem.

Despite the tacitness of Mathematish, the main part of mathematics teaching is by tradition calculation with formulas. The learners are heavily confronted with a "foreign" language in the mathematics classroom. Many learners perceive mathematics as a large set of fragments with an almost non-existing larger picture about adults mathematics memories of their school time (Lindenskov, 2001). This is a natural consequence when a general description of the language Mathematish is absent. Such a general description, showing similarities between isolated calculations, constitute a grammar. Furthermore, learners often feel unfamiliar with the very symbols they use when calculating – the alphabet of Mathematish. Rather than courses in Mathematish grammar, teacher-learner dialogues could be a good tool for formulating the relevant aspects of Mathematish. In a dialogue you can "play" the language game and detect the rules in social interaction (Wittgenstein, 1967).

Learning of foreign languages

The grammars of foreign languages that are learned later in life than a mother tongue are usually not tacit. Then the learning is done with the grammar of the language, which therefore is conscious. It is known that a language learned later in life is represented differently in the human brain than a mother tongue. Furthermore, Mathematish seems to be represented in the brain differently than natural languages (Butterworth, 1999). In an example, one person, after a brain damage, could not read "54" but could read "cinque quattro", which is Italian for "five four". Sometimes it is the other way around; one patient with brain damage could not read the phoneme based words signifying a specific number, but could read (and understand) the digits signifying the same number (*ibid*.).

The development of Mathematish started to a large extent as a short hand for mathematics expressed by natural language. An example is the Italian mathematicians who started in the fifteenth century to replace standard words such as *cosa* (the unknown thing), *censo* (square), and *radice* (root) with the abbreviations *c*, *ce* and *R*. Luca Pacioli replaced *pio* (plus) and *meno* (minus) with *p* and *m* with small horizontal lines above them (Katz, 1998). Another typical example of this change is the following cite from Robert Recorde in his introduction of the equality sign (Kaplan, 1960):

> And to avoide the tediouse repetition of these woordes 'is equalle to' I will sette as I doe often in woorke use, a paire of paralleles, or Gemowe [twin] lines if one lengthe, thus = because noe .2. thynges, can be moare equalle.

Unlike natural languages, Mathematish has been *written* from the start. Being born as a shorthand for natural languages, it naturally inherits some grammatical elements from natural languages, for instance logical variables. However, due to the specific use of Mathematish, which is quantitative calculations, it has a development which differs strongly from the development of natural languages.

Mathematish is usually encountered in elementary school, however most humans develop mathematical intuition earlier in life (Clements & Sarama, 2004; Heiberg Solem & Lie Reikerås, 2004). If mathematics intuition and Mathematish connect or stay separate for students is a central question for didactics of mathematics.

It appears as if Mathematish becomes intuitive and effective as a mother tongue for a rather small minority in the population. A basic educational problem for mathematics is that mathematics teachers often come from this group, while many of the students do not. Then many learners may have problems in mathematics of a kind that represent tacit knowledge for many teachers. We regard this as a problem that must be recognized fully in the entire mathematical community. This is particularly serious in the mathematics teacher education. An example of the present weak Mathematish awareness is that there is no general agreement about a very basic language question: what is a word in Mathematish?

Mathematish and computer programming

Mathematish has symbols that are concept based, as is the case of Chinese, and not phoneme based, as in the case of English. As a result, symbols and "words" may be pronounced differently in different parts of the world, however written essentially the same way. Hence there is no need for translating the symbols, a fact that facilitates communication and mathematics development. However, a demand for translation would force clarification of the structure of Mathematish and diminish its tacitness, as has been the case for natural languages.

Computers have been constructed with mathematics and logic as its basic structure. Computer programming languages have been developed which provide alternative ways of expressing mathematical ideas, algorithms and facts. The grammar is often similar to that of Mathematish. Some computer programming languages can partially be considered as dialects of Mathematish. This allows computers to effectuate formal mathematical calculation with no regard to meaning. It appears as if mathematics content cannot be expressed by computers, in the sense that the formal calculations appear to be very inefficient once there are no clear rules for how to calculate. This can be considered as a late endorsement by computer technology of Wittgenstein's claim that there cannot be rules for using rules in the same language.

The term "vernacular" is used for a native spoken tongue as opposed to constructed or official ones. The term "mathematical vernacular" was introduced by de Bruin in 1987 in a computer science context (de Bruin, 1987). The term has been established for a formal language for writing mathematical proofs that resembles the natural language from mathematical texts. There exist several such systems today, such as Hyperproof and Mizar. These are attempts to construct new languages or representation systems for increased consistency or efficiency, while Mathematish represents the present factual use of mathematical notations and symbols.

Mathematish-content interplay in mathematics

Two kinds of mathematical knowledge

In the previous example equation, $ax + by + c = 0$, five "unknowns" (letters) are present. Most mathematics teachers probably think of x and y as parameters, and a, b and c as constants, and a mental image of a straight line given by the values of the constants a, b and c may appear. This is not at all given by the equation itself. The geometric interpretation is an example of "mathematics content". You could also for instance interpret the equation merely as a relation between numbers. Such a concept of content is strongly culturally dependent and often personal. It is not easily formalized or defined, since it by definition is not formulas. As regards the meaning of mathematical content as knowledge that cannot be written in Mathematish, we may talk about content of two different kinds:

1. Mathematical meanings as the target of the symbols and expressions of Mathematish
2. Mathematical knowledge that cannot be expressed in Mathematish.

Examples of the first kind are applications of mathematics and geometrical figures that may be represented by formulas, where some may be rather personal. Further examples are concepts such as "oneness", "twoness", and so on, as properties of certain sets, represented by the symbols "1" and "2", and so on.

Examples of the second kind are strategies for problem solving, ideas of proofs and calculations, and evaluation of models and results (Ernest, 1999).

Very often mathematical equations are starting points of mathematical thinking, and mental "anchors" for various considerations of mathematically active persons. Many of these considerations are essential for successful mathematical work, however non-formalisable and partially personal. We consider also this as part of mathematics content of the second kind.

Semiotic approaches to Mathematish

As mentioned above our perspective opens for the use of methods from discourses like linguistics and semiotics, and we will use some terms and ideas from for instance de Saussure and Peirce, and their followers often named post-structuralists and neo-pragmatists.

From de Saussure we borrow the idea that a "sign" has two parts; the *signifier* and the *signified*. Saussure himself stressed that both signifier and signified were on a *mental* level, but in accordance with many post-Saussurians we stress the signifier as a *material* entity, for instance the ink doodles constituting a text in a book. The signified, though, we claim is a *concept* or a kind of mental picture. Although the signifier is treated by its users as "standing for" the signified, Saussure emphasizes that there is no necessary, intrinsic, direct or inevitable relationship between the signifier and the signified. The link between them is quite *arbitrary:* "the signs used in writing are arbitrary, the letter t, for instance, has no connection with the sound it denotes" (Saussure 1916/1983). The links, when culturally established, become parts of a structure, and the meaning of the signs is regulated by this structure and systematic relations between the signs. No sign makes sense on its own; the meaning of "tree" is related to other signs, for instance "bush". Saussure uses an analogy with chess, noting that the value of each piece depends on its position on the chessboard. While *signification* (what is signified) clearly depends on the relationship between the two parts of the sign, the *value* of a sign is determined by the relationships between the sign and other signs within the system as a whole (Saussure, 1983). The signifiers reflect *differences* that are important for the language users; the meaning of a sign is about *what it is not*, rather then what it is.

From Peirce we use the idea that the sense-making of a sign requires an act of *interpretation* and therefore an interpreter. The interpreter produces his own "sign" of the external sign in his mind, and this sign must also be interpreted. This model is sometimes called "the semiotic triangle" with the three parts sign vehicle, sense and referent.

The process of interpretation, the *semiosis,* could be ongoing in several steps, in principle ad infinitum. A very familiar situation where the signified also could play the role of signifier is when you are using a dictionary; sometimes also some terms in the defining text must be defined. The semiosis could take a dialogic form in one person's mind or between persons. While Saussure emphasizes structure in a synchronic way, Peirce emphasizes diachronical aspects. Peirce argued that "all thinking is dialogic in form. Your self of one instant appeals to your deeper self for his assent" (Peirce 1931-58). The same idea of *dialogical understanding* is elaborated more deeply in Bakhtin (1981).

Peirce also made a typology of signifiers, depending on the grade of their arbitrariness. *Symbols* are quite conventional and have to be learned, *icons* are in some way resembling the signified, and *indices* are directly connected, like photographs, measuring instruments, and indexical words (that, this, here, there).

The Saussurian concepts stress Mathematish as a ready-made cultural phenomenon with a given structure, while the Peircian concepts stress Mathematish learning and understanding as a subjective interpreting activity, both aspects of importance for our analysis. We will also use Wittgensteins concept of *language game (Sprachspiel)* to highlight the social aspect of Mathematish, and that "understanding" is to do the right thing in this "game", see (Wittgenstein, 1967).

There is also an ontological question about Mathematish that has bearing for the philosophy of mathematics. Umberto Eco says "a symbol is a lie" (Eco, 1976), i.e. it stands for something else, but what? This is one way to answer:

> "For example...the expression $x^2 + y^2 = 1$ can be seen as mixture of numbers and letters with no particular significance, as an algebraic equation, as a representation of a circle, or *as* a circle" (Brown, 2001, p. 193).

What is Mathematish *about*? Is it about objective existing concepts now labelled, is it about constructed objects now labelled, is it the “real” mathematics, is it a template for economizing thought, or is it perhaps just a sometimes useful game?

As we have seen in the history of mathematics, and also in our teaching practice, sometimes the notation creates the concept, and sometimes the other way around. This is also a theme in mathematic education research. For instance, Sfard describes how mathematical discourse and mathematical objects are creating each other in the learning process (Sfard, 2000), and how template-driven activities create concepts.

Content - beyond the concept

There are many forms of mathematical content. The content closest to Mathematish is the set of truths, i.e. the true statements that mathematicians consider to be true in the sense of being consequences of the axioms. This kind of content can appear almost indistinguishable from its Mathematish formulation (see amalgamation below), partially since Mathematish calculation is the dominant way that is used for checking its validity. This content is defined by Mathematish calculation.

Another part of mathematical content is images and associations connected to abstract entities. It is quite possible to give a strict definition of the number 2. But the digit “2” will also have personal connotations for a student. Part of this meaning is related to experiences of this particular quantity (“2”), perhaps from a multitude of examples (two apples, two ideas, two hands,…), and from a more intrinsic mathematical direction: from knowledge of even numbers and factorization of integers. This may be parts of a content “behind” the strict concept of number 2. A mathematical sign is therefore in practice signifying not only a strict mathematical concept but also (or instead!) a big amount of personal conceptions and memories, typical for the person reading or using the sign. This is usually referred to as *concept image* (Tall & Vinner, 1981).

Concept construction

The notion of Mathematish is useful when analysing the process of students' concept construction in cultural and cognitive aspects. As an example we chose the introduction of different kinds of numbers in school mathematics. We will analyse three aspects of this; *existence forcing, amalgamation* and *translation*. The analysis reflects the theory that a discourse constructs its objects and “reality” by introducing signifiers, and relates to Derrida (1976) and the elaborated ideas in Sfard (2000).

In everyday language a name like “table” could be introduced ostensibly (“look, this is a table”); you *point* at the signified object. Many aspects of language could be *shown* in practice and in interaction between language and action. In mathematics this is not possible, since visible objects are at most approximative examples of objects. Even in geometry the visible object is just a representation of the mathematical object: it is for instance hard to draw a line with no thickness. To discuss the nature of these objects is beyond the scope of this paper, but it is indeed an interesting ontological question. The pointing procedure must be substituted by something else. Also the *handling* of objects is invisible, and you cannot ostensibly *show* how to handle mathematical objects. You must use *material* signifiers for these purposes, for instance hands-on materials or the signifiers in Mathematish. Often operations with Mathematish are used to present and motivate new kinds of "names", for instance signifiers for numbers. In the following examples different kinds of numbers are presented by referring to Mathematish operations:

> "We have that 8 – 5 = 3 but what about 5 – 8?
>
> "We have that $\frac{15}{3} = 5$ but what about $\frac{3}{15}$?
>
> "We have that $x^2 = 4$ has the roots $x = 2$ or $x = -2$, but what about $x^2 = 3$?
>
> "We have that $x^2 - 4 = 0$ has the roots $x = 2$ or $x = -2$, but what about $x^2 + 4 = 0$?

In all these cases the operators used grammatically correct will provoke new kinds of results, and these results will in turn become signifiers for new objects. The result is transformed to a "name". Often you could still trace the operator in the signifier, for instance 3/7, a fact that is sometimes confusing for the learner: how could 3/7 and 9/21 be "the same number"? When using Mathematish grammatically correct new types of objects are *forced into existence*, often without a pre-existing learner intuition (if you are a Platonist this is of course not what is happening; instead Mathematish helps to "remember" the object). The construction of new concepts is not a process started in the learner's mind, on the contrary the language structure initiates and constructs the concepts. The concepts are not firstly existing, and than "baptized", on the contrary the names exist *before* the concepts (Wittgenstein, 1983). The same situation can be seen in the history of mathematics. For example, firstly the mathematicians constructed complex numbers by Mathematish rules, and later claimed that these should be seen as signifiers for a new kind of number (Katz, 1998).

A common tool in textbooks is to use the "number line" and put signifiers in a row along a line. Often textbooks say that "the negative numbers" are on this line, but in fact they are not. You could only see the ink doodles. Also in this case the very presentation of "names" will force objects into existence, according to Sfard. She points out that the introduction of new names and new signifiers is the beginning rather than the end of the story (Sfard, 2003). She demonstrates how the new signifiers for negative numbers appear from the very beginning: at the same time as the description of the new concepts.

A common problem in this existence forcing process is that the concepts desired never will start to exist in the learner's mind. Instead the signifier and the signified *amalgamate*; the signifier *is* the mathematical object, the object *is* the ink doodle. Mathematish may then for the learner appear as a "meaningless" procedural game with no relevance for intuitive thinking or outside school. *Translations* to mathematical prose and other types of representations are here very important to "help the object into existence". Pictures, analogies, metaphors and schemas can in a dialectical way interact with Mathematish results in order to strengthen the learner's intuition and creativity. Following Peirce, an interpretation of a signifier is an ongoing process, as a dialogue, and this dialogue is necessary also between Mathematish and mathematical prose.

A problem with many textbooks is that they are in fact encouraging amalgamation: "the *line* $y = 3x + 4$", "the *function* $y = x^2 + 3x$" and so on. These expressions in Mathematish are not presented as special representations of mathematical objects, but as the very objects themselves.

Interesting questions are for instance what the difference is between a mathematical fantasy, or "lie", and a mathematical concept forced into existence, and why students believe (or should believe) in these concepts.

Mathematics produces Mathematish rules – and vice versa

Mathematish consists of pure conventions, and of rules of calculations. Examples of pure conventions are the choice of symbols, such as "=" for equality instead of "#" or "EQ", or choices of notation such as writing a^n for n factors of *a*, and not $^n a$, *a_*^{n} or *pow*(*a*,*n*) (e.g. Bergsten, 1990; Pimm, 1987).

Logic and other truths are often formalized into rules of calculation: a *calculus*. These rules may then be used without any regard to their meaning. Examples are $x(y+z) = xy + xz$ (i.e. replacing $x(y+z)$ by $xy + xz$ is OK), $0 = 3 - 3$, $\sin(\arcsin x) = x$. Which rule is meaningful at a particular instance depends entirely on the goal and purpose of the calculations; hence on the mathematics content. Such rules of calculation take the form of grammatical rules. A counterpart in English is the statement "The horse pulls the car" that can be replaced by "The car is pulled by the horse. Hence, developments in mathematics give new Mathematish rules to use.

But sometimes it is the other way around; a calculation with mere Mathematish creates an unexpected result that afterwards has to be interpreted. This holds both for school mathematics and research.

Mathematical intuition – a human trait

A part of becoming human is learning to handle quantity, size, space and order – practical forms of mathematics that often are not formulated in Mathematish. The process of learning to walk is strongly driven by instinct, but also involves and develops the mind. Simultaneously, consciousness of the body and of three-dimensional geometry develops. One may say that every human develops mathematical intuition from the first years in life, which is formulated more or less verbally. When beginning school, this intuition meets the official language of mathematics: Mathematish.

During the first years in school, mathematics only concerns the symbols +, =, –, ·, / and the ten numerals. These symbols are certainly abstract. The abstraction lies in the generality: the same symbols are used for counting or measuring *anything*. This generality can be seen as the most basic property of the nature of mathematics: a separate formulation for calculation that is independent of application areas, and effective for all of them. It also represents the main leap of thought that challenges pupils.

Philosophy and practice

The identification of a special mathematics language may seem to be a rather philosophical endeavour, but in this paper we have tried to show that philosophy and classroom practice go hand in hand. The basis of observations about students' relations to Mathematish is our teaching practice, the teaching practice of our colleagues, and findings in mathematics education. We have described mathematics as a personal mathematics intuitive content that is both expressed and shaped by elaborated mathematics notations, called "Mathematish". We hope that our perspective, that Mathematish is a complete *language*, could create fruitful analogies with other languages.

References

Alexandersson, C. (1985). Stabilitet och förändring. En empirisk studie av förhållandet mellan skolkunskap och vardagsvetande. *Göteborg Studies in Educational Sciences, 53*. Göteborg: Acta Universitatis Gothoburgensis.

Bakhtin, M. (1981). *The dialogic imagination: Four essays* (trans. C. Emerson and M. Holmquist). Austin, TX: University of Texas Press.

Bergsten, C. (1990). Matematisk operativitet: en analys av relationen mellan form och innehåll i skolmatematik. (Mathematical operativity: An analysis of the form-content relationship in school mathematics) *Linköping Studies in Education. Dissertations No. 29*. Linköping University.

Bergsten, C. (1999). From sense to symbol sense. In I. Schwank (Ed.), *European research in mathematics education I.II* (pp. 126-137). Forschungsinstitut für Mathematikdidaktik, Osnabrück.

Brown, T. (2001). *Mathematics education and language. Interpreting hermeneutics and post-structuralism.* Revised 2nd Edition. Dordrecht: Kluwer Academic Publishers.

de Bruijn, N.G. (1987). The mathematical vernacular, a language for mathematics with typed sets. In P. Dybjer et al., (Eds.), *Proceedings of the Workshop on programming languages*. Marstrand, Sweden.

Butterworth, B. (1999). *The mathematical brain.* London: Macmillan.

Clements, D. H. & Sarama, J. (2004). *Engaging young children in mathematics.* Mawah, NJ: Lawrence Erlbaum Associates.

Davies, P.J. & Hersh, R., (1980). *The mathematical experience.* Boston: Birkhäuser.

Derrida, J. (1976) *Of grammatology* (Spivak Trans.) London: John Hopkins Press.

Eco, U. (1976). *A theory of semiotics.* Bloomington: Indiana University Press.

Engström, A. (2002). Semiotik och matematiddiaktik. En introduktion. Arbetsrapporter vid Pedagogiska institutionen, Örebro universitet.

Ernest, P. (1999). Forms of knowledge in mathematics and mathematics Education: Philosophical and rhetorical perspectives. *Educational Studies in Mathematics, 38*, 67-83.

Foucault, M. (1961). *L´histoire de la folie à l´age classique. Folie et deraison.* Paris.

Heiberg Solem, I. & Lie Reikerås, E. K. (2004). *Det matematiska barnet.* Stockholm: Natur och Kultur.

Hjelmslev, L. (1961). *Prolegomena to a theory of language.* (trans. Francis J. Whitfield) Madison: University of Wisconsin Press.

Kaplan, E., (1960). *Robert Recorde (c. 1510-1558): studies in the life and works of a Tudor scientist.* PhD thesis, New York University.

Katz, V. J. (1998). *A history of mathematics. An introduction.* Addison-Wesley.

Kline, M. (1980). *Mathematics. The loss of certainty.* New York: Oxford University Press.

Lindenskov, L. (2001). Vanskeligheder i relationen mellem matematik og virkelighed – plevt som meningslöshed. I M. Niss (ed.) *Matematikken og verden* (s. 195-217). Köpenhamn: Fremad.

Löthman, A. (1992) *Om matematikundervisning – innehåll, innebörd och tillämpning.* (Uppsala Studies in Education, 40). Uppsala: Acta Universitatis Upsaliensis.

Marion, M. (1998). *Wittgenstein, finitism, and the foundations of mathematics.* Oxford: Clarendon Press.

Molander, B. (1996). *Kunskap i handling.* Göteborg: Daidalos.

Nunes, T., Schliemann, A.D. & Carraher, D.W. (1993). *Street mathematics and school mathematics.* New York: Cambridge University Press.

Peirce, C.S. (1931-58). *Collected writings.* (8 vols.) (eds. Charles Hartshorne, Paul Weiss and Arthur W. Burks). Cambridge, MA: Harvard University Press.

Polanyi, M. (1967). *The tacit dimension*, New York: Anchor Books.

Radford, L. (2000). Signs and meanings in students´ emergent algebraic thinking: a semiotic analysis. *Educational Studies in Mathematics, 42*, 237-268.

Pimm, D. (1987). *Speaking mathematically: Communications in the mathematics classroom.* London: Routledge and Keagan Paul.

Ryle, G. (1949). *The concept of mind*. Chicago: University of Chicago Press.

Saussure, F. de (1916/1983). *Course in general linguistics* (trans. Roy Harris). London: Dockworth.

Sfard, A. (2000). Symbolizing mathematical reality into being: How mathematical discourse and mathematical objects create each other. In P. Cobb, K.E., Yackel & K. McClain (Eds.), *Symbolizing and communicating in mathematics classroom: Perspectives on mathematical discourse, tools, and instructional design.* Mahwah: Erlbaum.

Sfard, A. (2003). There is more to discourse than meets the ears: Looking at thinking as communication to learn about mathematical learning. In C. Bergsten & B. Grevholm (Eds.), *Challenges in mathematics education. Proceedings of MADIF3* (pp. 34-49). Linköping: SMDF.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics*, *12*,151-169.

Waismann, F. (1979). *Ludwig Wittgenstein and the Vienna circle.* From the notes of F.Waismann, Oxford: Blackwell.

Winsløv, C. (submitted). Semiotics as an analytic tool for the didactics of mathematics. Paper submitted to Nordic Studies in Mathematics Education.

Wittgenstein, L. (1967). *Philosophische Untersuchungen*. Frankfurt am Main: Suhrkamp Verlag.

Wittgenstein, L. (1983). *Remarks on the foundations of mathematics*. Blackwell, Oxford.