
FROM SENSE TO SYMBOL SENSE

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Abstract: *In this theoretical paper figurative aspects of algebraic symbolism are discussed and related to the theory of image schemata, opening up for one way of understanding the development of symbol sense in mathematics. The ideas of mathematical forms (referring to spatial characteristics of mathematical formulas), and form operations, are at the core of this analysis.*

Keywords: -

1. Introduction

The term "symbol sense" has only recently come into use in mathematics education, referring to a similar kind of familiarity with algebraic symbols as "number sense" to arithmetic: "As students' understanding of algebra deepens, they are gaining *symbol sense*: an appreciation for the power of symbolic thinking, an understanding of when and why to apply it, and a feel for mathematical structure. Symbol sense is a level of mathematical literacy beyond number sense, which it subsumes." (Picciotto & Wah 1993, p. 42) With a developed symbol sense as the main goal of algebra teaching (Arcavi 1994), a theoretical framework of the conception and its development is needed. The issue is complicated by the fact that algebra can be viewed both as a symbol system and as a way of thinking (cf Sierpinska 1995, p. 157). The interest here is on aspects of the symbolic language used in school algebra.

Leibniz, one of the most influential innovators of notations in the history of mathematics, wrote (see Cajori 1929, p. 184): "In signs one observes an advantage in discovery which is greatest when they express the exact nature of a thing briefly and, as it were, picture it; then indeed the labour is wonderfully diminished." This quotation (also quoted in Bergsten 1990), referring to notations in mathematics, indicates that when symbols in mathematical formulas are arranged in a way similar to those structures of the real world that the formulas are used to depict, then the notation is easy

to understand and to work with. The idea that mathematical thinking, as well as understanding, as related to formulas, can be facilitated by observing the figurative characteristics of formulas, can find further support in the image schemata theory of Mark Johnson and George Lakoff, as noted by Dörfler (1991). In this theoretical paper some conceptions of mathematical form, introduced in Bergsten (1990), and in line with these ideas, are further developed and related to symbol sense. It should be pointed out that the space available for this conference paper does not allow a deeper elaboration of the ideas presented.

2. Mathematical Forms

When looking at strings or expressions of (standard) mathematical symbols (in elementary school mathematics) as pictures, only a very few basic patterns (spatial arrangements) appear. Bergsten (1990) thus defined *forms* as typographical units (*atomic forms* or *symbols*) or as spatial relations between typographical units (*molecular forms* or *patterns*). The notion of *mathematical forms* refers to symbols used in mathematics and patterns of such symbols. When combining atomic mathematical forms like 1, 2 and 3 to produce mathematical expressions like 213, 1+2, or 1+2=3, the molecular forms appear as the spatial characteristics (schematic structures) of the symbolic expressions. The forms here can be depicted by the schematic structures of the figures *III*, *IOI*, and *IOIOI* respectively. Seeing 1+2 as a unit (chunking) in the last of the expressions, the essential form is seen to be *IOI* there as well. This is the form of a *link* (that can be extended to a *chain* like in *IOIOI*), used in notations of both operational (1+2) and relational (1<2) ideas. The other molecular form that appeared here (i.e. *III*) has the pattern of *bars*. An algebraic expression like $3x + y = 5$ is a molecular form structured by the following “Chinese boxes”: $(((3 \cdot x) + (y)) = (5))$. It can be observed that the pattern is composed by a number of links only, in this case three.

Also when operating on mathematical symbols some basic schematic structures appear. Some of these are *splitting* (as in $3x \rightarrow x+2x$), *joining* ($x+2x \rightarrow 3x$), *mirroring* or exchange of position ($b+a \rightarrow a+b$), *adding* ($3 \rightarrow 30$ when multiplying by ten), *deleting* ($30 \rightarrow 3$ when dividing by ten), *raising* ($2x \rightarrow x^2$ when integrating), *lowering*

($x^2 \rightarrow 2x$ when differentiating). The schematic structures describing these symbol transformations can thus be seen as form operations and are composed into *form operators* (transforming for example an equation like $7x-3=4x+6$ into its solution $x=3$). Also the *identity* operation can be identified as a form operation. The role of form operators in algebra is essential. Pimm (1995) even argues that "the algebra takes place between the successive written expressions and is not the statements themselves." (p. 89)

Some basic forms seem to have their origin in bodily experience. These forms can be called *genetic* (Bergsten 1990), by which is meant that they can be isomorphically mapped onto pictures (or objects) of possible referents of the symbolic expressions. A simple example is provided by the expression $2+3$, the form of which (link) also appears in the spatial arrangement of the picture $oo\ ooo$, showing the act or the idea of adding together two and three objects, a possible referent of the symbolic expression. The form of the algebraic counterpart $a+b$ is inherited from the world of arithmetic, thus bringing along its genetic character. Forms that are not genetic are called *stipulated*, as most atomic forms (like 2 or $\sqrt{\quad}$) or the form *superposition* used in exponential notation (as in x^2 or e^x).

Observations like these are of course trivial but seem to have far reaching implications for the understanding of (some parts of) mathematical symbolism:

- The three molecular mathematical forms just mentioned are not only commonplace, but practically the only ones that are used in (standard) elementary school mathematics to formalize the ideas of numbers and the four basic operations of arithmetic (forms for executing some algorithms excluded). They are also (notably as a consequence) commonplace in algebraic notation (including abstract algebra and vector algebra).
- The observation that genetic forms seem to be based on bodily experience invites the use of image schemata theory in discussing understanding and meaning of (some parts of) mathematical symbolism. In particular, the individual development of (some aspects of) symbol sense can be traced back to the structure of abstract thinking ultimately based on sense impressions.

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- A structure-operation distinction within mathematical symbolism, based on the above definitions, provides a means of analysing the understanding of mathematical forms, using the complementary concept.
 - The ability to link form and content in mathematics is commonly considered a main feature of what it means to understand mathematics (see e.g. Hiebert 1986). The dynamic interplay between these four dimensions of mathematics and doing mathematics (i.e. form and content, and structure and operation) gives a characteristic of the logical and psychological aspects of mathematics (Bergsten 1990).
 - Bearing in mind that all discussions of learning and understanding mathematics in educational contexts by necessity must rest on some view of what mathematics is, the conception of mathematics as a *formalization* of ideas originated in our bodily experience or in other disciplines (like physics or economics), as elaborated by Mac Lane (1986), puts an emphasis on mathematical forms, as well as on its relation to the corresponding ideas and activities, when it comes to understanding.

The notion of genetic forms is related to the idea of analogical reasoning (Gentner 1989, English 1997), in the algebra context discussed in English and Sharry (1996). It is commonly argued that these "surface" structures of mathematical symbolism are mere conventions, and that the meaning they carry must be found by the individual's own mental constructions of their relationship to the "deep" structure of mathematics, or the algebraic manipulations become meaningless (English & Sharry 1996). Here it is argued, however, that meaning often can be more or less directly evoked already at the "surface" level by its very structure.

It should be observed that the notion of mathematical forms does not extend to a possible more abstract notion of "mathematical form", in the sense of general structures that are being formalized as mathematics. Such structures can to some extent be expressed by non-algebraic forms of representation, such as verbal and pictorial forms. The discussion in this paper is focussed on the implication for the understanding of school algebra of the notions of mathematical forms and image schemata.

3. Mathematics and Image Schemata

The conceptual framework of image schemata by Johnson (1987) and Lakoff (1987) has a tremendous potential impact on mathematics education. Its relevance to meaning in mathematics and mathematical reasoning and understanding has been pointed out by the authors themselves, and later, among others, by Presmeg (1991) and Dörfler (1991).

According to Johnson and Lakoff, meaning and understanding in human reasoning are based on the use of *image schemata*. Our bodily experience of ourselves and our physical environment, and our way of structuring and organizing this experience, establish cognitive image schemata (with a non-propositional character) that depict recurring regularities and structures of these experiences and organizing activities. These schemata thus are constructed by the individual. Examples of some basic image schemata are CONTAINER, BALANCE, PATH, LINK, SCALE, AND CENTER-PERIPHERY (see e.g. Johnson 1987, p. 21, 85, 113, 117, 121, and 124, respectively). A key notion in the theory is that these kinds of bodily based schematic structures also are used in human abstract thinking by the means of metaphorical projection from the world of bodily experience into the abstract dimension.

In her study on visualization in mathematical thinking Presmeg (1985) found two ways students used imagery to depict abstract situations, by *concretizing the referent* (“making a concrete visual image the bearer of abstract information”), and by *pattern imagery* (“embodies the essence of structure without detail”). The latter is a construct similar to image schemata (Presmeg 1991, p. 7).

Dörfler (1991), stressing the holistic aspect of (subjective) meaning of mathematical concepts, argues that the (mental) construction of appropriate image schemata (referring to Johnson and Lakoff) is one means of facilitating the “cognitive manipulation” of such concepts. This construction can be supported by working with protocols of actions.

According to Mac Lane (1986) activities like counting, ordering, shaping, and moving, give ideas of the common structures and regularities in these activities (across situations), which can be formalized as mathematics. Thus the activity of counting gives the idea of “the next one”, which can be formalized as “successor” by using the

Peano axioms (giving a formalization of the mathematical “object” ordinal number). Behind mathematical concepts like limit and continuity is the idea of approximation, originated in activities like estimating. More complex mathematical structures seem to be generated from within mathematics itself. Thus it is appropriate to talk about *content-dominated* mathematics and *form-dominated* mathematics (Bergsten 1990), thereby indicating that the mathematical forms used in the former have a more direct link to ideas based on bodily experience, or, in the above terminology, are predominantly genetic.


Many mathematical notions can be related to ideas of an image-schematic character. Set theory can be seen as the formalization of properties of the CONTAINER schema, distance as the CENTER-PERIPHERY and SCALE schemata, a function as the LINK schema and its graph as the PATH schema, and an equation as the BALANCE schema. Many concepts and methods of mathematics seem to be developed from the human way of thinking by the use of bodily based image schemata. Whatever the “real” mechanisms behind the historical development of mathematical notions and notations have been, the use of image schemata theory when discussing mathematics seems to have a high potential value in the educational context. Dörfler writes (1991, p. 20): “The cognitive manipulation of mathematical concepts is highly facilitated by the mental construction and availability of adequate image schemata. In other words, the subjective meaning of mathematical terms has a non-verbal, non-propositional and geometric-objective component. The individual understanding of a mathematical topic possibly is best grasped as a kind of interplay between the propositional expressions and corresponding image schemata.” The last sentence from this quotation can possibly be seen as an image-schematic interpretation of the above mentioned (common) view that much of mathematical understanding relies on making connections between form and content. The (subjective) meaning of mathematical content then is given by its image schematic structure, a view in line with Mac Lane’s description of mathematics, as indicated by Lakoff (1987, p. 365, referring to an article by Mac Lane from 1981): “Mac Lane’s view of mathematics is thus very much like the view of human conceptual systems that has emerged in this book.” The forms of mathematics “are those that emerge from our bodily functioning in the world and which are used cognitively to comprehend experience.” (Lakoff 1987, p. 365)

These ideas have recently been further developed by Lakoff and Núñez (1997), who set up the new task of describing the "metaphorical structure of mathematics" (p. 21), and they write: "One of the properties of commonplace conceptual metaphors is that they preserve forms of inference by preserving image-schema structure." (p. 30)

4. Mathematical Symbolism and Image Schemata

The relevance of the image schemata theory to (the understanding of) mathematical symbolism has been noted by Dörfler (1991, p. 28): "I think that formulas in mathematics can play the role of a carrier for an appropriate image schema. The latter then is made up by the spatial relations of the symbols in the formula and of the admissible operations and transformations with the formula. It is this image schema which lends meaning to the formula as its concrete carrier". Here similar structural (figurative) and operational aspects of mathematical symbolism as those introduced in Bergsten (1990), and discussed above (in the paragraph on mathematical forms), are stressed.

With this focus on the figurative aspects of mathematical symbolism, a key observation is that genetic mathematical forms (as defined above) show the same schematic structures as the BALANCE schema, the LINK schema and others, and this almost by definition! That forms (schematic structures) are genetic means that they are based on the way humans structure and organize their bodily experience. A possible redefinition of the notion *genetic* mathematical form, keeping the intention behind the original definition intact, relates such a form to a (similarly structured) prototypical image schema behind the idea of the referent of the symbolic expression. Now, image schemata are subjective, but this redefinition makes sense, since many basic schemata are prototypical and socially shared.

When an idea is based on the activity of putting things together, it (by human imagination) can be mentally structured by the LINK schema, by Johnson (1987, p. 113) schematically materialized by a picture like . That the recording in some notational form of such an activity or action (what Dörfler calls protocol of an action), shows the same schematic structure as the image schema itself, is more likely to be expected than not. The historical success of the mathematical form used in the standard

notation for addition, $a+b$, thus has a psychological explanation in its prototypical image schematic character.

Johnson (1987, p. 90 and pp. 95-98) suggests that the BALANCE schema, with its properties of symmetry, transitivity, and reflection, provides a possible experiential basis for the ideas behind the mathematical concepts equality and equivalence relation. In the schematic structure of the symbolism, the equality sign (in itself made up by two equal parts, as stressed by Robert Recorde himself) serves as the fulcrum of the balance. In the terminology above, the mathematical form of written equalities is a link (when chunking of left and right “wings” is made), and must be classified as genetic by its connection to the BALANCE schema.

Form-dominated mathematics naturally inherits some of its mathematical forms from the content-dominated mathematics, making it possible to “understand” such forms by metaphorical projection from the image schematic structures of genetic forms. The notation for associativity of compositions of elements of groups, a purely formal property, is (most likely) inherited from the notation for repeated addition (or multiplication), exhibiting a genetic mathematical form. The ease of operating on such purely formal expressions, as well as the sense of understanding (in some way) what one is doing, can be explained by the image schemata that are being evoked by observing the schematic structures of the formal expressions.

Dörfler’s (1991) recommendation to use protocols of actions (to promote the construction of adequate image schemata) to develop understanding of mathematical concepts genetically related to those actions, thus also applies to develop understanding of mathematical forms. The common element is the construction of image schemata. In the case of forms, however, the actions of interest are the reading, manipulation, and interpretation, of (mathematical) symbolic expressions.

The notion of mathematical forms brings the focus to the structural (figurative) aspects of (mathematical) symbolic expressions. Form operations deal with the operative aspect of symbolic expressions, transforming them to new symbolic expressions. Some common such operations have been identified above, and their image schematic character is based on bodily experience. No doubt, activities like splitting or joining (putting together), adding or deleting (taking away), and raising or

lowering, are prototypical across many experiential situations from low ages on. Also the identity operation can be experienced, as when turning a cube upside down or rotating a ball, leaving the object “unchanged”. It makes sense to refer to some form operations as *genetic*, as splitting, joining, and mirroring, on some symbolic expressions in arithmetic (see examples above), while others, like adding, deleting, raising, and lowering (as exemplified above), are *stipulated*. Using Dörfler’s (1991) terminology, image schemata related to mathematical forms are *figurative*, image schemata related to form operations are *operative*.

5. Developmental Aspects of Understanding Forms

Real numbers are used (“applied”) in many different kinds of settings: though dealing with a variety of situations (denoting magnitudes, proportions or time) they still obey the same formal rules. To “understand” what real numbers are then by necessity involves “understanding” the formal rules for such numbers. These formal rules are “visible” in mathematical formulas such as $a+b=b+a$. This means that for a learner, understanding of mathematical forms (in the sense defined above) is a path that (possibly) can lead to an improved understanding of mathematics. Indeed, genetic forms like in the above formula (for commutativity) have the potential of evoking the same image schemata as the ideas or activities they depict. This way a link between form and content can be (mentally) established, and the idea of understanding formulas as “pictures” of image schemata can be evoked. To operate on formulas of genetic form is like operating directly on the reality. They are isomorphic activities. This observation may be one key to answering the classical question on why mathematics works, i.e. that results of symbolic manipulations are applicable to reality (Bergsten 1990, p. 166; cf e.g. Kline 1985).

Thus, in mathematical understanding there is a dynamic interplay between form and content, facilitated by the use of image schemata. On the other hand, there is an interplay between structural and operational aspects of form as well as of content. In her analysis of conceptual development, Sfard (1991) stresses the complementarity of the structural and operational aspects of mathematical concepts and entities. Understanding the structural aspect of a concept presupposes the use of operational

aspects of concepts with a structure already understood. However, there is a vicious circle: “*the lower-level reification and the higher-level interiorization are prerequisite for each other!*” (Sfard 1991, p. 31). This can explain, Sfard argues, why for many students the whole enterprise of mathematics can become a meaningless game of rules producing correct answers. The reason is that reification has not occurred. Seeing a structure is more difficult than performing an operation. Therefore many students need well-planned educational activities for the reification process. The use of protocols of actions to produce adequate image schemata (Dörfler 1991), has the potential of facilitating this process.

Understanding mathematical forms seems to develop in a similar way. The schematic structure of a formula can become visible by operating on instances where the formula holds. For the “reification” of the form (of the formal notation) for commutativity, for example, adding different pairs of numbers in both orders and systematically recording these operations symbolically (e.g. in columns), the schematic structure of these notations can be interiorized, performed and “seen” purely mentally, and finally reified as a formal structure. In this case the known form used is the *link* form of the addition formula, and the new form produced the form operation *mirroring* for addition formulas. The sense of understanding these genetic forms is hypothesized to be given by the similarly structured image schemata they evoke, originated in bodily based activities and experiences.

Now, there is no need to understand the *formula* for commutativity (not to be mixed with the *idea* that adding in reverse order yields the same result) until it must be used in a formal computation. In school this normally happens only after algebraic notation has been introduced, and that is also when understanding mathematical forms is beginning to become crucial also for mathematical achievement. For some students the algebraic terms themselves are void of meaning: they are simply symbols (atomic forms) like a , b , and c , in written (rule-governed) expressions like $a+b=c$. Thus in beginning algebra there is a *double meaninglessness*: operations are performed on algebraic expressions, made up by terms without meaning, and with a form without meaning.

To develop an understanding of mathematical forms, and pave the way for symbol sense, its structural and operational aspects must both be grasped. These seem to presuppose each other, i.e. are complementary for this understanding. To apply the form

operation joining on the left wing of the expression $7x-x=54$, to obtain $6x=54$, the schematic structure of the expression must be understood (possibly by the LINK and/or BALANCE image schemata). On the other hand, to see the basic structure, the link around the equality sign, it must be observed that a joining operation can be performed on $7x-x$.

6. Symbol Sense

Some aspects of symbol sense in the context of school algebra have been described by Arcavi (1994), one of which is strongly related to the present discussion: "[What is symbol sense?] - An ability to manipulate and to "read" symbolic expressions as two complementary aspects of solving algebraic problems." (p. 31) It is argued in this paper that the development of this (key) aspect of symbol sense can be facilitated by building on the recognition of the role of image schemata and genetic mathematical forms for the feeling of understanding the mappings between form and content in school algebra. These constructs provide a link from sense impressions to the development of symbol sense in elementary school algebra.

Higher-level symbol use in (form-dominated) mathematics rests heavily on familiarity with the basic algebraic language used in school. The role of non-propositional, imagistic thinking is furthermore prevalent at all levels of mathematical thinking (cf Sfard 1994).

7. References

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