"I NEED THE TEACHER TO TELL ME IF I AM RIGHT OR WRONG"

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This talk presents some thoughts about the possible reasons of students' tendency to rely on teachers for the validity of their solutions and of their lack of sensitivity to contradictions in mathematics. Epistemological, cognitive, affective, didactic and institutional reasons are considered in turn.

SOME FACTS TO EXPLAIN THE CONTEXT OF THE QUOTATION IN THE TITLE

The facts come from a research on sources of frustration in adult students of pre-university level mathematics courses required by a university for admission into academic programs such as psychology, engineering or commerce (Sierpinska, 2006; Sierpinska et al., 2007).

Fact 1. In a questionnaire[1] used in this research there was the following item:

I need the teacher to tell me if I am right or wrong.
Agree Disagree Neutral

Of the 96 students who responded to the questionnaire, 67% checked "Agree".

Fact 2. Six respondents were interviewed. One of them, female, about 21 years old, a candidate for admission into commerce, was required to take a calculus course. She failed the first time round. She re-took the course in the summer term and passed, but found the whole experience extremely frustrating. Here is what she told us, among others:

> *My teacher in the summer, he was a great teacher, he explained well and everything, but it's just that I could never grasp, like I couldn't be comfortable enough to sit down in front of an example and do it on my own, instead of looking back at my notes. Okay, what rule was it and why did I do this? I just never understood the logic behind it, even though he was a great teacher, he gave us all possible examples and he used very simple words…, start from the very easy and try to add things on to make it more difficult. But… how I studied for the final? I was looking at the past finals… All by memorizing, that's how I passed [this course] the second time.*

Fact 3. In items 74 and 75 of the questionnaire, students were asked for their preference regarding two kinds of solutions, labelled "a" and "b", to two inequalities with absolute value ($|2x-1|<5$ and $|2x-1|>5$, respectively). Solutions "a" could be called "procedural"; they are commonly taught in high schools and consist in reducing the solution of an inequality to solving two equations and then following certain rules to write the final solution to the inequality. Solutions "b" resembled those taught in introductory undergraduate courses focused on logic; they referred explicitly to properties of absolute value and use logical deduction. In item 75, solution "a" ended with an incorrect answer: a condition on *x* which contradicted the initial inequality. Nevertheless, in both items, there was a clear preference for solutions "a" (69% chose this solution in item 74 and 62% in item 75). Only about 1/5 of the respondents chose solutions "b" in each item. The choice of these solutions was almost always justified by reasons other than correctness: "clearer", "easier", "simpler".

Fact 4. Four instructors of the prerequisite course were interviewed. All reported students' dislike of theory and proofs and preference for worked out examples of typical examination questions. They said that students prefer to memorize more rules and formulas than to understand how some of them can be logically deduced from others and memorize fewer of them. They reported eventually giving in to students' preference and avoiding theory and proofs in their classes.

One of the instructors (female, PhD student) told us:

> *[Students don't want to reason from definitions about] those rules, [although] all the rules come from the definition (…) It's especially true when we learn (…), the seven rules of exponentia[tion]. Sometimes, I just try to let them know that [it is enough to just] know four [rules], or even three, if one knows the definition well. You don't need to put so much time on recalling all those rules in your mind. But when I try to explain those things, they don't like it. They ask me 'Why, why you do this?'*

Another instructor (male, PhD student) was telling us that he would do very little theory in class, replacing proofs by graphical representations, and giving significance to theorems, formulas and methods by using historical anecdotes:

> *Actually, to be honest, I don't do much theories, or proof or anything like this, you know, in such classes. I try to avoid it as much as I can, but let's say about the integral thing that I just did, I filled out all the rectangles with colours, and then I told them this is fun, then I put the definition of definite integral, then I put a remark: 'In fact this is a theorem, you know... and it was proven by Riemann... '. Then I told them about Riemann a little bit, they were happy, and that he still has problems and it's worth many dollars to solve this. So I made the mood and then I moved to fundamental theorem of calculus, just the statement without proof, without anything you know, then start giving examples. (…) I don't think they like proofs because in a proof you cannot put numbers or anything, you have to do it abstractly and this they hate. Yeah, they don't like this.*

EDUCATIONAL VS ECONOMICAL GOALS OF MATHEMATICS TEACHING

Why teach mathematics? According to Ernest (2000), the answer depends on who is speaking. From the perspective of economic theories of education, mathematics may be seen as contributing to general purposes of education such as,

- Building human capital by teaching skills that directly enhance productivity;
- Providing a screening mechanism that identifies ability;
- Building social capital by instilling common norms of behaviour, and
- Providing consumption good that is valued for its own sake.

(Gradstein et al., 2005: 3).

Most educationists eschew using such pecuniary terms when they speak of the purposes of education although they do realize that words such as "capital" and "consumption good" might better reflect the current reality than their ideals. After all, they do invoke financial issues when they discuss the reasons of the difficulty of achieving their preferred goals of education, or deplore the tendency of some universities to become "'corporate entities', where students are 'clients' and traditional values – 'raising the better-informed citizen' – are losing ground to job-training" (Curran, 2007).

Educationists prefer to continue viewing the purpose of education as "to provide rich and significant experiences in the major aspects of living, so directed as to promote the fullest possible realization of personal potentialities, and the most effective participation in a democratic society", based on "reflective thinking", with mathematics contributing to its development by providing the person and citizen with analytic tools especially appropriate for dealing with "quantitative data and relationships of space and form" (Committee on the Function of Mathematics in General Education, 1938: 43-45)[2].

Is actual education in general, and mathematics education in particular, anywhere near achieving this lofty goal? Actual mathematics teaching is often blamed to foster rote learning of computational and algebraic techniques, geometric formulas and textbook proofs, thus failing to contribute to the education of critical citizens and reflective thinkers. This leads to reform movements and curriculum changes. But consecutive reforms don't seem to change much. For example, the lament over rote learning was used both in promoting the famous "New Math" reforms of the 1960s and in their criticism later on (Kline, 1973; Freudenthal, 1963; 1973; Thom, 1970; 1972; Chevallard, 1985).

EFFORTS AT ACHIEVING A CONCEPTUAL MODEL OF MATHEMATICS TEACHING

Since the (in)famous “New Math” reforms, there has been relentless theoretical and experimental work on the design and study of classroom situations engaging students in mathematical thinking and reasoning (e.g. Brousseau, 1997). The emphasis on independent, creative and critical mathematical thinking and mathematical reasoning in educational research and ideology appears to have made it to curriculum development, if only in the form of rhetoric. But in some countries, classroom activities aimed at these goals have been institutionalized or are prepared to be institutionalized (e.g. in Québec[3]). The “situational problems” start with a general description of a situation (intra- or extra-mathematical) supposed to provoke students to formulate their own questions, propose solutions and defend them in small groups and whole classroom discussions. Observers of mathematics classrooms where such activities take place are usually duly impressed by students’ engagement. Still, in-depth analyses of students’ productions and the content of teacher-student interactions bring disappointment as to the level of critical and autonomous mathematical reasoning actually done by the students (e.g., Brousseau & Gibel, 2005).

The constructivist movement in North America tried to eliminate the rote model by a fundamental change in teachers’ and educators’ epistemology of mathematics. This resulted in bitter “math wars” (Schoenfeld, 2004) but rather not in the desired epistemological changes. Representatives of the opposed camps speak at cross purposes: one side attempts to prove that schools with more traditional curricula and methods of teaching produce better scores on standardized tests (Hook, 2007), while the other argues that knowledge developed in reformed schools cannot be measured by such tests.

Reformers (of constructivist or other profession) in North America are, however, far from saying that replacing the rote model by the conceptual model is easy. It is even hard to convince some people that it is necessary, especially if the rote model “works”, in the sense described by Goldin in the quote below.

> At all socioeconomic levels, [the US] society persists in setting low educational goals. In wealthy, suburban communities, where the intellectual and physical resources for quality education are generally available, there is a disturbing tendency for schools to coast, particularly in mathematics and science. Here it is easy for school administrators to cite high achievement levels, evidenced by standardized test scores and students’ admissions to prestigious universities, as hallmarks of their schools’ successes – although these may be due more to the high socioeconomic status of parents than to high quality education. Why push our children if they are already doing fine?... Why take risks, when bureaucracy and politics reward stability and predictability? (Goldin, 1993: 3).

What doesn't work, according to Goldin, is the teaching of mathematics. Maybe something else is being taught but not mathematics, which is conceptual knowledge:

> For students to go beyond one- or two-step problems in mathematics requires conceptual understanding, not the ability to perform memorized operations in sequence; in removing the development of this understanding from the curriculum, *we have removed the foundation on which mathematics is built*. (Goldin, 1993: 3; my emphasis)

However, the conditions formulated by the author for the replacement of the rote model by the conceptual model appear very costly in terms of funding, organized human effort, and cultural changes that would also take a long time to stabilize.

> The [reform] initiatives that have been undertaken must be increased drastically if we are to arrive at a new cultural context – one in which elementary and secondary school teachers have seen in some depth pure and applied mathematical research, and move easily in the university and in industry; one in which research mathematicians and scientists know some of the problems of education, and move easily in schools; a context based on one large community of mathematical and scientific researchers and educators, rather than the disjointed groups we have now. (Goldin, 1993: 5)

THE CONTEXT OF MY CONCERN WITH THE CONCEPTUAL MODEL OF MATHEMATICS TEACHING

I became interested in this problem because of the facts mentioned at the beginning of this paper: the apparent tendency, among students of the prerequisite mathematics courses to rely on teachers for the validity of their solutions and to remain insensitive to even obvious mathematical inconsistencies; and approaches to teaching that seem to enhance these attitudes. Similar phenomena were observed by other researchers in other groups of students (Lester et al., 1989; Schoenfeld, 1989; Stodolsky et al., 1991; Evans, 2000; FitzSimons & Godden, 2000).

These are disturbing results from the point of view of the educational goals of teaching mathematics that we cherish. In the case of the prerequisite mathematics courses, it seems even hypocritical to force candidates to take these courses by telling them that they will need the mathematical theory and techniques in their target academic programs, and then fail to even develop their independence as critical users of mathematical models. Do we have any use for financial advisers who are not critical with respect to the predictive mathematical models they are using and blind to the mistakes they are making? How credible are reports of psychologists who use statistical methods in their studies but do not understand the theoretical assumptions and limited applicability of the methods they are using? Is it necessary to mention engineers who design wobbly pedestrian bridges because they fail to notice that the computer program they were using for their design assumed only vertical and not lateral vibrations? (Noss, 2001).

The prerequisite courses certainly serve the purposes of academic selection in the administrative and economical sense of reducing the number of candidates to such levels as the human and material resources of the respective university departments are capable of handling. These courses are also a source of financial support for the mathematics departments who staff them with instructors, markers and tutors, recruited from among faculty, visiting professors and graduate students. Their existence is, therefore, institutionally guaranteed.

The question is if it is possible to make these courses serve educational as well as administrative and economical purposes, by modifying the teaching approaches and convincing students of their value for their future study and professions. *Realistically* possible, that is, which means respecting the constraints under which these course function. They must be short and intensive because students are adults who may have jobs and families: they cannot spend a lot of time in class and they can't wait to have the prerequisites behind them and start studying the core courses of their target programs. Classes are large and there is pressure to make them even larger, for economical reasons; universities are always short of money. There is also the lack of professional pedagogical knowledge or experience among the instructors (graduate students, professors), who, when they were university students themselves, have rarely if ever experienced any other form of teaching than a lecture, occasionally interrupted by questions from the students or short problem solving periods. At the university, in mathematics departments, there is no pressure and certainly no requirement to teach otherwise. This may not be the most effective method of teaching but it is the least costly one in terms of intellectual and emotional effort. Graduate students of mathematics are not experienced and confident enough, neither in mathematics, nor in classroom management skills (not to mention language skills), to conduct an investigation or a mathematical discussion. Professors are usually more interested in a neat organization and smooth presentation of the mathematical content than in knowing what and how students in their class think about it. Indeed, they may not want to know, for fear of losing morale. It is more pleasant to live in the illusion that students think exactly the way we think ourselves. Grading tests and examinations is usually a rude awakening, which depresses teachers for a little while. But it is better to be depressed just for a short while than all the time, realizing in every class that whatever one says is understood by the students in a myriad of strange ways, most of which have nothing to do with the intended mathematical meaning.

I had the following hypothesis, which I probably shared with many of fellow mathematics educators. If students in the prerequisite courses were lectured not only on rules, formulas and techniques of solving standard questions but also on some of the theoretical underpinnings of these, then they would have more control over the validity of their solutions and would be more interested in the correctness of their solutions. Knowing the reasons behind the rules and techniques would allow them to develop a sense of ownership of mathematical knowledge. Teachers and students would be able to act more like partners in front of a common task. There would be a possibility of a discussion between the teacher and the student about the mathematical truth. If the student only follows the teacher's instructions, discussion of mathematical truth is replaced by the verdict of an authority: the teacher decides if the student is right or wrong. The theoretical discourse would distance the student from his or her self-perception as someone who either satisfies the expectations of an authority, or stands corrected. There would be no reason for the student to delegate all responsibility for the validity of his or her solutions to the teacher. Moreover, since justified knowledge is more open to change and adaptation in dealing with novel situations, it is more easily transferable to other domains of study and practice and not good only for solving the typical examination questions (Morf, 1994) and therefore more relevant; it is worth teaching and learning.

A teaching experiment

I planned to use a teaching experiment to explore this hypothesis: a mathematical subject (I chose inequalities with absolute value, to keep the same topic as in the research on frustration) would be presented in a short lecture using different approaches, some stressing effective procedures for solving a type of problems, others - the underlying theory. Subjects (recruited from among students of the prerequisite mathematics courses) would then be asked to solve a series of problems (the same for all groups). The experiment would end with a “task-based interview” (Goldin, 1998), where students would be asked questions about their solutions to the given problems, and about their views and habits relative to checking the correctness of their solutions. In particular, they would be asked questions such as: how do you know this is a correct answer? When you solve assignments or do test questions, how do you know you are right? Are you even interested in knowing this? I was hoping to see if there is any relationship between the teaching approach used and students’ control over the correctness of their results.

At the time of writing this paper, only 13 students have been interviewed. It is too early to draw conclusions. But the results, so far, suggest a far more complex reality than my naïve hypothesis had it. Students have many reasons for checking or not checking their answers and they do it in a variety of ways. Students following the theoretical approach lectures were not clearly more likely to care about the validity of their answers; in fact, after the procedural lectures, more students seemed concerned with the validity of their answers than after the theoretical approach lectures.

Why could it be so that the teaching approach doesn't matter so much? In the rest of this paper I am going to offer some hypotheses about the possible reasons for this state of affairs, which state is still quite hypothetical, of course, but also made more plausible by the hypotheses.

More specifically, I will be talking about the possible reasons of students' dependence on teachers for the validity of their solutions (abbreviated "DT") and their lack of sensitivity to contradictions ("LSC").

It is difficult to find a single theory that would explain the DT and LSC phenomena although there have been attempts in educational research to capture as much as possible of the complexity of teaching and learning (e.g., Illeris, 2004; Chevallard, 1999). My reflection will therefore be eclectic, borrowing ideas from a variety of theoretical perspectives. I will organize it along the following categories of possible reasons: epistemological, cognitive, affective, didactic, and institutional.

EPISTEMOLOGICAL REASONS

[DT] *Much of mathematics is tacit knowledge*. Dependence on the teacher might be something that is specific to mathematics not *in fact*, but *in principle*. Essential aspects of mathematical ideas and methods cannot be made explicit (Polanyi, 1963). It is difficult to learn mathematics from a book. There are non-verbalized techniques that are learned by interacting with a master; doing a little and getting quick feedback. There is a lot of implicit schema building for reasoning and not just information-absorbing and deriving new information directly by association or simple deduction. (Castela, 2004).

[DT] *A mathematical concept is like a banyan tree*. The meaning of even the most basic mathematical concepts is based on their links with sometimes very advanced ideas and applications that are not accessible to the learner all at once and especially not right after having seen a definition, a few examples and properties. Initial understanding is necessarily fraught with partial conceptions, over- or under-generalizations or attribution of irrelevant properties (some of which might qualify as epistemological obstacles; Sierpinska, 1994). The student is quite justified in feeling uncertain about his or her notions and in looking up to the teacher for guidance.

[LSC] *Contradiction depends on meaning*. Consider the expression: "$-x < 2$ and $x < -2$". This expression represents a contradiction if it is understood as representing a conjunction of two conditions on the real variable *x*. There is no contradiction if the second term of the expression is understood as the result of an application of the rule "if $a < b$ and $c \neq 0$ then $a/c < b/c$" to the first term of the expression, understood as an abstract alphanumeric string, and not as an order condition on a real variable. The rule is a "theorem-in-action" (Vergnaud, 1998: 232) that seems to be part of many students' mathematical practice.

[LSC] *Contradiction presumes there is meaning*. The above assumption that contradiction depends on meaning implies that there *is* meaning, that is, if a statement is meaningless for you, the question of its consistency does not exist for you.

Let us take the example of absolute value of a real number. For the mathematician, the absolute value may be associated with situations where only the magnitude – and not the direction – of a change in a one-dimensional variable is being evaluated. It may thus be seen as a particular norm, namely the two-norm in the one-dimensional real vector space, R^1: $|x| = \sqrt{x^2}$. This notion makes sense only if numbers are understood as representing the direction and not only the magnitude of a change, i.e. if "number" refers to both positive and negative numbers. Absolute value is an abstraction from the sign of the number. If "number" refers to magnitude, it has no sign and there is nothing to abstract from. Moreover, the notion is useless if only statements about concrete numbers are considered; it offers a handy and concise notation only when generality is to be expressed and algebraically processed (for a historical study of the concept of absolute value, see Gagatsis & Thomaidis, 1994). The commonly used definition

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

encapsulates all these meanings, of course, but doesn't make them explicit. For a student with a restricted notion of number and un-developed sense of generality in mathematics, this definition is meaningless. It is therefore not surprising that he or she does not see contradictions in statements such as $|x| = \pm x$ or $|x+1| < -3|x-1|$ (for more information on and analyses of students' mistakes in the domain of absolute values, see Chiarugi et al., 1990; Gagatsis & Thomaidis, 1994).

[LSC] *Contradiction requires rigour in definitions and reasoning*. "Contradiction" applies to statements where the meaning of terms is stable in time and space. Thus it applies to rigorous texts whose discursive function is closer to *objectivation* of knowledge rather than its *communication* (Duval, 1995; see also Sierpinska, 2005). But the function of mathematics textbooks, at least at the pre-university or undergraduate level is communication, not objectivation. In such textbooks, the boundaries between definitions and metaphors, illustrations and proofs, are often blurred.

The aim of teaching at those levels is to help students develop "some sense" of the concepts and a few basic technical skills, with the hope that, if needed, the concepts will be reviewed in a more rigorous manner at the graduate level, for those who will choose to study mathematics for its own sake. Even the fathers of mathematical rigour in Analysis as we know it today, Bolzano (1817/1980) and Dedekind (1872/1963), conceded that too much concern for rigour and proofs in the early stages of its teaching would be misplaced.

To illustrate the confusion between definitions and metaphors in didactic texts, let us look at the introduction of the notion of absolute value in a college level algebra textbook (Stewart et al., 1996: 17). The section "Absolute value and distance" starts with a figure (below) and the following text:

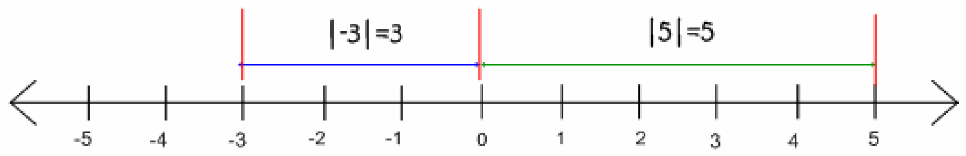

Figure 1. Reproduction of "Figure 9" in (Stewart et al., 1996: 17)

> The absolute value of a number *a*, denoted by |*a*|, is the distance from *a* to 0 on the real number line (see Figure 9). Distance is always positive or zero, so we have |a| ≥ 0 for every number a. Remembering that – a is positive when a is negative, we have the following definition [the two-case formula is given next in a separate paragraph which is also centered and bordered]. (Stewart et al., 1996: 17)

The first sentence of the text reads as a definition; it has the syntax of one. Yet it is not one because it uses the term "distance" as a term borrowed from everyday language, and thus as a metaphor. Distance in everyday language means something different than in mathematics, where it is assumed that the meaning of this word is fully determined by the following three properties and only these properties: 1. distance from point A to point B is the same as distance from point B to point A (so that the orientation of the movement between A and B is ignored); 2. that distance is not the path but a measure of the path and that this measure is an abstract number and not the number of centimeters or inches or other units (i.e. that it is a ratio); and 3. that going from A to B and then from B to C we cover a distance that is not less than the distance from A to C (so that $|a+b| \leq |a| + |b|$ can appear obvious later on). If, for the reader, distance is the number of units, and not a pure number, then, at this point, he or she may well think that $|-3| = 1.5$ cm and $|5| = 2.5$ cm according to the accompanying figure. Of course, the annotations on Figure 9 aim at eliminating this ambiguity: they suggest that the distance of a point representing a number on the number line is to be measured in the unit chosen for this representation and not in some other units, and that the distance is the number and not a number of units. Thus a lot of information is contained in the figure if only one knows what to look at in the figure.

Grasping the intended meaning of this text requires also certain conceptualizations that it may be unrealistic to assume in the readers.

One is the correspondence between numbers and points on the number line, which is not an easy concept (see, e.g., Zaslavsky et al., 2002). In everyday life, distance refers to places in space, not to numbers, so talking about distance between numbers doesn't make sense. Yet, this correspondence is taken for granted in the text: in the first occurrence of the symbol “a”, it refers to a number; in the third, without warning – to a point.

Another assumption is the algebraic understanding of the symbol $-a$. The conception that this symbol represents a negative number is well entrenched in students, even at the university level (Chiarugi et al., 1990).

COGNITIVE REASONS

[LSC] *Sensitivity to contradictions in mathematics requires theoretical thinking*. (Sierpinska & Nnadozie, 2001; Sierpinska et al., 2002 – Chapter I, section, “Theoretical thinking is concerned with internal coherence of conceptual systems”). Theoretical thinking is not a common mental activity; it is not the first one we engage in when confronted with a problematic situation. When the situation is mathematical, theoretical thinking may be common among mathematicians but not among students.

The object of theoretical thinking is an abstraction from the immediate spatial, temporal and social contexts; these contexts, on the other hand, are in the centre of attention in practical thinking. Questions such as, *Is this statement true? Is it consistent relative to the given conceptual system?* make sense from the perspective of the theoretical mind, but not necessarily from that of the practical mind. Here, it is more natural to ask, *Does this technique work?, Is this answer good enough? Is this argument sufficiently clear, convincing, acceptable, under the circumstances?* The practical thinker is oriented towards acting in the situation, solving the problems at hand with the available means; he or she does not reflect on the various interpretations of the situation, the hypothetical solutions and their logically possible consequences.

For the action-oriented student, obtaining correct answers is guaranteed by "doing what one is supposed to do" according to examples provided by the instructor or a book.

Let me illustrate this point with a story from my teaching experiment.

Student AD (female, 21-25 years of age, candidate for a major in mathematics and statistics) was following a procedural approach lecture on solving inequalities with absolute value. She then solved six exercises, the second and third of which were, respectively, $|x-1| < |x+1|$ and $|x+3| < -3|x-1|$.

In both she followed the procedure shown on the example of $|x-1| < |x+2|$ in the lecture. Her solutions are reproduced in the *Appendix*. She obtained a correct solution in exercise 2 but her solution to exercise 3 was "∞" by which she meant that the inequality holds for all real numbers. She did not check her solutions by plugging in concrete numbers into the initial inequalities and so was unaware of the mistake in exercise 3. The interview included the following exchange:

> AS: Tell me how you did the second one (|*x*-1|<|*x*+1|)
>
> AD: I just did it the same way as they did here. I solved for zero on both sides.
>
> AS: You solved for *x*?
>
> AD: I put *x*-1=0, *x*+1=0, so *x*=1, *x*=-1, and then I put it in a chart and I solved if *x* is smaller than -1, if it's in between and greater than 1. And I solved it in these three cases. In the first case, both are negative, which means I put negative in front, before the bracket and I came out with that.
>
> AS: How do you know you are right?
>
> AD: Because that's how you are supposed to do it? I don't know!

In the interview about exercise 3, the student was encouraged to verify if the inequality is true for some concrete numbers, like 1, 2, -1. When she found that the result is false in each case, her reaction was, "I don't know what I did wrong". She was not satisfied until she found where she failed in applying the procedure. The object of her thinking was not the internal consistency in a set of mathematical statements, but her actions in relation to a task.

[LSC] *Noticing a contradiction in conditions on variables is harder than in a statement about concrete numbers*. Some students never miss a contradiction in statements such as -1 > 2, but have no qualms about "simplifying" the condition "$-x < 2$" to "$x < -2$". In the teaching experiment (procedural approach), one of the students (LA, male, over 30 y.o., applying for admission into computer science), would start by numerical testing of the given inequalities, and only when he arrived at exercise 4, he looked back at the notes from the lecture and attempted algebraic processing. He never made any mistakes in judging the validity of his numerical statements. With regard to the inequality $|x+3| < -3|x-1|$, he started by testing it for -3, 1, 2, and 3, always getting a contradiction. Then he looked up the lecture notes, and, as he said in the interview, he "finally understood what he was supposed to do". So he engaged in "analysing cases". His algebraic work was full of mistakes. There were careless mistakes (e.g., re-writing the right-hand side of the inequality as "*x*+1" instead of "*x*-1"; dividing 6 by 4 and getting 2/3, etc.). There were systematic mistakes such as not changing the direction of the inequality when dividing it by a negative number, and logical mistakes (not taking into account all possible cases and ignoring the conditions on *x* defining the intervals in which each case would be valid). His conclusion was $x < -\frac{2}{3}$. He crossed out his first numerical calculations and left the algebraic nonsense as his final answer.

[LSC] *Noticing a contradiction in a longer mathematical message is demanding on cognitive functions* such as attention, information processing, and memory, especially "mathematical memory" which doesn't seem to be particularly common among students and is considered to be a gift (Krutetskii, 1976). These cognitive faculties appeared rare among the subjects of our teaching experiment. Some admitted that their minds "wandered away" during the lecture and they missed some essential points. In fact, only one student (YG, female, 26-30 y.o. candidate to commerce) became completely absorbed in listening to the lecture (procedural approach), so that nothing mathematically essential for the presented method escaped her attention. She was the only one, among the 13 students, whose solutions to all exercises were complete and correct. She told us she tested her algebraic solutions with numerical calculations but did not bother writing these calculations down. She also told us that, in the mathematics courses, "most of the time I can understand the stuff during class. Actually, I seldom do questions after class. I use time during the class efficiently. So after class, at home, I seldom do mathematics. But just before examinations, I will study sometimes".

AFFECTIVE REASONS

[DT] [LSC] *The school mathematics discourse* (Moschkovich, 2007) *uses expressions such as "right" and "wrong" rather than "true" or false",* normally reserved for courses in logic. But "right" and "wrong" are emotionally laden, especially when uttered in relation with a student's work and not – mathematical statements independently from who had said or written them.

They are an element of an assessment and it is the teacher's job to assess, not the student's.

[LSC] *Relying on gut feeling about having got the "right answer*". Several students in our experiment mentioned "feeling" when asked, "How do you know you're right? Even YG mentioned feeling, although she also checked her answers by numerical substitutions. But this appeared to be "double-checking", not the primary or only checking. Here is an excerpt from our interview with her:

AS: When you solve a problem, how do you know you're right?

YG: (Silence)

AS: Are you making sure it's good?

YG: I think when you do the mathematic problem and you are on the right track, you have the feeling that you're on the right track.

AS: (…) Many people have the feeling they're right yet they get their answers wrong.

YG: Sometimes, when it's wrong, you will get a conflict, a contradiction. Actually, when I do some problems, I want to compare the answers, to make sure I'm right.

AS: Here, when you were doing these problems did you check your final answers?

YG: Yes, I would choose a number in this area and test, check the answer.

AS: You did that?

YG: Yeah. In heart, not write down.

[LSC] [DT] *Not verifying one's answers for fear of losing morale*. Here is what one student (BK, male, below 21, applying for admission into mechanical engineering), told us after having listened to a theoretical approach lecture and solved the 6 exercises. He solved half of them correctly. Each algebraic solution was followed by numerical substitutions for two numbers, but it turned out that the student did not attribute the status of verification to these calculations. He believed that these calculations were part of the expected solution. Contradictions between his numerical calculations and his algebraic work went unnoticed. He explained why he doesn't check his answers in the following terms:

AS: When you do your mathematics assignments, how do you know you are right?

BK: If doing it was smooth// if it was a smooth process, like I didn't find it was difficult, or// it was just flowing. Anyway, I never go back to check.

AS: Here, you may have felt that everything was going smoothly even if it was a little bit tedious, but still, not all your answers were right.

BK: Whenever I check and I realize I got it wrong, I start losing the morale. So I'd rather finish and then, if I feel I need to check, then I check. But if I checked in the middle and found a mistake it would have

> affected the way I was doing the other problems. (…) I just tend to believe I got everything right. I'd rather just receive the paper and be told what I got right and what I got wrong. I'm like, okay, I did the best I could. But if I'm at number 6 and I know I did the first three wrong then I start doubting the others, and I would not be happy after the test, I'd just walk sad, and I won't even want to receive my paper back. So I'd rather just not know.

DIDACTIC REASONS

[DT] *In didactic situations, the task is given by the teacher* and the decision if it has been satisfactorily completed is the teacher's responsibility; such are the rules of the *didactic contract* (Brousseau, 1997). The student's job is to produce answers, to the best of his or her knowledge. Under this contract, the "verification" or "check" part of working on an equation or inequality that the teacher demonstrates before the students does not have the function of reducing uncertainty, because the teacher is assumed to know the correct answer. It may appear to the students as part of a "model solution text" (as in the case of BK above). Some students, however, like YG, are able to see the epistemological difference between a solution and its verification. This student, while not indifferent to the rules of the didactic contract – she told us she wasn't sure if writing only an answer she had figured out mentally without "showing all her work" was acceptable – did not see it necessary to write down the numerical checking she did in her head.

[DT] *There are many tasks in school mathematics where it may be impossible or difficult for the student to verify the answer*. Ironically, proofs belong to this category. Students may be able to notice a blunt inconsistency, but they may not suspect the existence of a counterexample to one of their "theorems-in-action" if they don't know enough theory yet. Moreover, how much detail a proof should contain is a rather arbitrary decision and we have students asking us such questions as, "may I assume known that $\sqrt{p}$ is irrational for *p* prime, or do I have to prove from scratch that $\sqrt{19}$ is irrational in this particular exercise?" We normally proceed by local and not global deduction in presenting the material to students, and it is not always clear what can be assumed as proved, known, and what must be proved in a given problem. But even in research mathematics, the decision whether a proof is correct or not belongs to a group of experts; there is always a possibility that the author has overlooked an inconsistency. The completeness of a proof submitted for publication is decided by reviewers and editors and depends on the standards of rigour and detail of the particular journal.

[LSC] *Depriving students of opportunities for noticing a contradiction for the sake of "fairness" of assessment*. In the interviews with students following the teaching experiments, a few told us that they realize they made a mistake when problems are linked together so that the answer to the next depends on the answer to the previous one and they get something unexpected in the next one. They may not be able to elaborate on their reasons for knowing what to expect, (as in the transcript below) because this requires a meta-reflection on one's thinking processes and they may have no linguistic means to express themselves). However, to have expectations about the result of one's mathematical work must be based on some theoretical knowledge (even if it is based on theorems-in-action that are not all consistent with the conventional mathematical theory). The excerpt below comes from the interview with SC (male, less than 21 y.o., candidate to computer science), after a theoretical approach lecture.

AS: What methods do you use to check? How do you know you are right?

SC: Sometimes the equations are linked together, and when you get the wrong answer, you will not get the result expected in the second answer.

AS: But how do you know what to expect?

SC: (Silence)

AS: Can you give us an example of such a situation?

SC: Most of the time it appears when you are doing derivatives. We use the derivative to, uh, there is something we use, we use the derivative (pause). But the questions are linked, and if you don't get the good derivative, you have some problems to find the, the maximum or some other derivatives (pause).

AP: You get some contradiction in the table where you put intervals of the function increasing, decreasing, no?

SC: Yes, yes, exactly!

This points to the benefits of the didactic organization of exercises into interrelated sequences so that a mistake made in one exercise produces nonsense in the others and may give motivation for checking an answer. However, in marked assignments or on exams, for institutional reasons, questions are disconnected, so that the answer to the next exercise does not depend on the answer to the previous question. It is considered "not fair" to the students to link questions like that, so that mistakes carry over.

INSTITUTIONAL REASONS

[LSC] [DT] *In school, validity = compliance with institutional rules and norms*. In school practice, mathematics becomes, in fact, a collection (often a loose collection) of types of tasks (exercises, test questions, etc.) with their respective techniques of solution, where the form of presentation (e.g., "in two columns") often has the same status as the mathematical validity. Techniques are justified on the basis of their acceptability by the school authorities, not on their grounding in an explicit mathematical theory. It is not truth that matters but respect of the rules and norms of the didactic contract related to solving types of problems.

In the context of absolute values, school mathematics (in some countries) had developed a whole praxeology (in the sense of Chevallard, 1999), with specialized monographs on the subject for the use of teachers, where tasks were codified into types, methods of their solution exposed and justified internally relative to the definition of absolute value, without regard to the uses of the notion in domains of mathematics other than school algebra (Gagatsis and Thomaidis, 1994). In the process of didactic organization of the material for classroom teaching, some elements of the theoretical justification would inevitably disappear as too advanced for the students, or appear in the curriculum in ways that would eliminate their use as means of validity control (for examples of this phenomenon in the context of teaching elements of mathematical analysis, see Barbé et al., 2005).

FINAL REMARKS

In mathematics education we commonly blame students' poor knowledge of mathematics and negative attitudes to its study on procedural approaches to mathematics teaching and we claim that mathematics taught that way is not worth teaching or learning. We constantly call for reforms that would support conceptual approaches to mathematics teaching. I, at least, blamed students' dependence on teachers for the validity of their solutions and their lack of sensitivity to contradiction on the “rote model”. But what guarantee is there that those ills would be removed by adopting the conceptual model? A lot of money and human effort could be spent on implementing the desired model and the results might be quite disappointing. The expected students' interest, autonomy and mathematical competence might not materialize not because of lack of teachers' competence or good will but because of epistemological, cognitive, affective, didactic and institutional reasons that are independent of their knowledge and good will. These reasons have their roots in the nature of mathematics, in human nature, in the very definition of a didactic situation and in what makes a school a school rather than a Montessori kindergarten.

Perhaps conceptual learning can occur and actually occurs also in procedural approaches? In my modest experiment, 3 out of the 5 students who followed a procedural approach checked their answers; only one out of the 8 students who followed the theoretical approach did so. The only student, who checked her answers effectively and had all her solutions correct, followed the lecture with the procedural approach. But this student (YG) did not apply the method showed on an example in the lecture uncritically in her solutions. She was gaining new experience as she was solving the problems and was finding useful shortcuts. After having applied the taught procedure to the inequality $|x+3| < -3|x-1|$, she realized that it wasn't necessary because the inequality contained an obvious contradiction. After she solved $|2x-1| < 5$, she knew what to expect in $|2x-1| > 5$ and its solution only served as a verification of the first one. In the interview after the experiment she described what kind of teaching approaches she considers "effective". Below, I give an excerpt from the interview where she describes her experience. I will close this paper with this student's words. They are worth thinking about.

AS: The lecture you listened to, was it very different from what you are used to?

YG: I think it is almost the same. In class the professor also give you some example. They just, they didn't explain you the theory, they just give you example: "after the example you will understand what I am telling you, the definition".

AS: So there is no theory, just "here is an example; here is how you solve it".

YG: I also think this is an effective way. Actually, earlier this semester I met – I think it was a MATH 209 professor – his way of teaching was totally different from other mathematics teachers. He put more emphasis on explaining the theory. And the most strange thing was that he'd write down everything in words. Normally, in mathematics, the teacher never writes the words, just the symbols, but he wrote everything in words. So in class we just took down the words. It was like a book, a lot of words to explain. And I don't think this way of teaching is good for me, so I changed to another professor. (…) I think mathematics is not literature.

AS: Was it the writing that was bothering you, or the theory that he was using, justifying everything. What was it that you didn't like, words or the theory?

YG: No, we spent most of the time, just writing, not writing, copying, so you don't have the time to think and to understand. So that's what was not good.

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3. I thank the undergraduate students who volunteered to participate in the experiment as students.

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Appendix

Exercise 2. Reproduction of student AD's solution

$|x - 1| < |x + 1|$

$x - 1 = 0$ $\quad$ $x + 1 = 0$

$x = 1$ $\quad$ $x = -1$

	-1	1	
-	-	+	$x - 1$
-	+	+	$x + 1$
I	II	III	

Case I $\quad -x + 1 < -x - 1 \quad 1 < -1 \quad \varnothing$

Case II $-x + 1 < x + 1$ $0 < 2x$ $0 < x \leq 1$

Case III $x - 1 < x + 1$ $-1 < 1$ $x > 1$

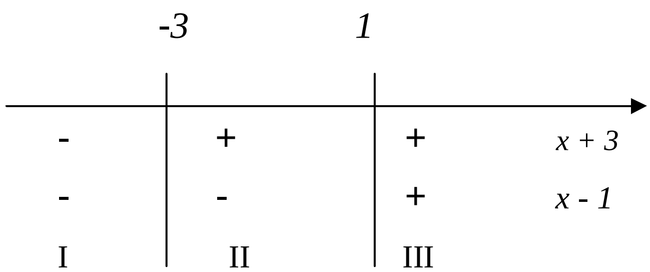
$| x - 1 | < |x + 1 | \Rightarrow x > 0$

Exercise 3. Reproduction of student AD's solution

$|x + 3| < -3 | x - 1|$

$x + 3 = 0$ $x - 1 = 0$

$x = -3$ $x = 1$



Case I $-(x + 3) < +3 (x - 1)$ $-x -3 < 3x - 3$ $x > 0$

Case II $x + 3 < 3x - 3$ $0 < 2x$ $-3 > x > 3$

Case III $x + 3 < -3x + 3$ $0 < -4x$ $x < 0$

$| x + 3 | < -3 |x - 1 | \Rightarrow \infty$

Endnotes

[1] The questionnaire, together with raw frequencies of responses, can be viewed at http://www.asjdomain.ca/frequencies_table.html

[2] The notion of "democracy" invoked by the quoted Committee was based on Dewey's (1937: 238) description: "Democracy… means voluntary choice, based on an intelligence that is the outcome of free association and communication with others. It means a way of living together in which mutual and free consultation rule instead of force, and in which cooperation instead of brutal competition is the law of life; a social order in which all the forces that make for friendship, beauty, and knowledge are cherished in order that each individual may become what he, and he alone, is capable of becoming."

[3] See, e.g., the document, "The Québec Education Program – Secondary Education", where the basic goals of teaching mathematics are stated as the development of the following three transversal competencies: 1. To solve a

situational problem. 2. To use mathematical reasoning. 3. To communicate by using mathematical language. The document is available at http://www.learnquebec.ca/en/content/reform/qep/ .