Stability and regularization of a backward parabolic PDE with variable coefficients *

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Abstract

We consider a backward parabolic partial differential equation with variable coefficient $a(x, t)$ in time. A new modification is used on the logarithmic convexity method to obtain a conditional stability estimate. Based on a formal solution, we reveal the essence of the ill-posedness and propose a simple regularization method. Moreover, we apply the regularization method to two representative cases. The results of both theoretical and numerical performance show the validity of our method.

1 Introduction

General second-order parabolic equations describe in physical applications the time-evolution of the density of some quantity u, say a chemical concentration, within the region Ω. The second-order term describes diffusion, the first-order term describes transport, and zeroth-order term describes creation or depletion (see [11]). The initial/boundary-value problem of the general second-order parabolic equations has been investigated in detail in [11]. However, in many situations, it is also necessary to recover the initial data, from some measurement given at $t = T > 0$. This is the so-called the backward parabolic partial differential equation in time (BPDE).

There has been much research work in these areas, such as [1, 2, 16]. However, in [1, 2, 16], the coefficients of the second-order term do not depend on the time t. In this paper, we are interested in the following BPDE:

$$\begin{cases} u_t = \nabla \cdot (a(x,t)\nabla u), & x \in \Omega, t \in [0,T], \\ u(x,T) = \varphi(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, t \in [0,T], \end{cases} \tag{1.1}$$

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where the domain $\Omega \subset \mathbb{R}^n$ is a bounded, open, connected set, and we want to get the data $u(x, 0) = \psi(x)$. The coefficient is assumed to be differentiable for every t and satisfy

$$0 < A_1 \leq a(x,t) \leq A_2, \tag{1.2}$$

and

$$a_t(x,t) \leq A_3, \quad A_3 > 0. \tag{1.3}$$

Many papers are devoted to special cases of problem (1.1). For instance, [5, 12, 15] investigated regularization methods and computation for the case $u_t = a u_{xx}$ with constant $a > 0$, which is called the backward heat conduction problem (BHCP), and [22] gave a stable numerical algorithm for the case $u_t = (a(x)u_x)_x$.

In general, no solution, which satisfies the heat conduction equation and the final data, exists. Further, even if a solution did exist, it would not be continuously dependent on the final data [19].

Unfortunately, some techniques, such as the methods of separation of variables and fundamental solution, used for the BHCP cannot be easily applied to problem (1.1). To the author's knowledge, so far there are few results about problem (1.1). Such as (a) Under the assumption that there exists solutions, is the solution unique? (b) Compared with some other ill-posed problems, is the degree of the ill-posedness strong or weak? (c) Compared with the BHCP, what is the difference and the similarity? (d) How to use the regularization technique to it for both the theoretical and numerical aspects? Therefore, it is the purpose of this paper to investigate problem (1.1) and try to answer the above four questions.

We give answers to the above four questions in subsequent sections. In section 2, to resolve question (a), we use the logarithmic convexity method to obtain a conditional stability estimate and uniqueness of solution. In section 3, we obtain a formal solution (3.11), from which we reveal the ill-posedness in essence and construct a simple regularization method. Many regularization methods should be used, see [20] etc. But there are some serious competitors when it comes to numerical algorithms. Furthermore, we compare problem (1.1) with some well-known ill-posed problems including (1) high order numerical differentiation [21], (2) an inverse spacewise-dependent heat source problem [23], (3) a Cauchy problem for the Laplace's equation [9], (4) inverse heat conduction problem (IHCP) [20], and (5) BHCP [5]. After some analysis, problem (1.1) is found to be the most strong ill-posed and is similar in some ways to the BHCP [12]. Section 4 describes the numerical implementation for general domain. Some numerical examples are presented in Section 5 and concluding remarks can be found in Section 6.

2 Conditional stability estimate

Conditional stability results are very important for both the direct and inverse problems. In the past years, many papers were devoted to the conditional stability of PDEs. As early as 1961, Miranker gave a stability result for a BHCP in [17]. Actually, the conditional stability results imply not only the uniqueness of the solutions of the problems (see Corollary 2.6) but also the convergence rate of the regularized solutions ([6]). Among many methods to obtain the conditional stability, Carleman estimates (e.g.[14]) and the logarithmic convexity method (e.g.[1, 8, 18]) are two main ones. Here we use the logarithmic convexity method to obtain a stability result of Hölder type.

Let $u(\cdot,t)\in H_0^1(\Omega)$ be a solution satisfying system (1.1) in the weak sense, where $H_0^1(\Omega) := \{w(x)|Dw\,\text{exists in the weak sense and belongs to}L^2(\Omega), w = 0\,\text{on}\,\partial\Omega\}$. We define an energy function $F(t)$ for the function $u(x,t)$ as follows:

$$F(t)=\int_\Omega u^2(x,t)dx. \tag{2.1}$$

Therefore

$$F'(t)=2\int_\Omega u(x,t)u_t(x,t)dx. \tag{2.2}$$

Together with system (1.1), using integration-by-parts and the boundary conditions, we also have

$$F'(t)=2\int_\Omega u\nabla\cdot(a(x,t)\nabla u)dx=-2\int_\Omega a(x,t)\nabla u\cdot\nabla u dx, \tag{2.3}$$

and

$$F''(t)=-2\int_\Omega a_t(x,t)\nabla u\cdot\nabla u dx-4\int_\Omega a(x,t)\nabla u\cdot\nabla u_t dx. \tag{2.4}$$

Using integration-by-parts again, we have

$$\int_\Omega a(x,t)\nabla u\cdot\nabla u_t dx=-\int_\Omega u_t\nabla\cdot(a(x,t)\nabla u)dx, \tag{2.5}$$

which implies

$$\int_\Omega a(x,t)\nabla u\cdot\nabla u_t dx=-\int_\Omega u_t^2 dx. \tag{2.6}$$

Inserting formula (2.6) into (2.4), we get

$$F''(t)=-2\int_\Omega a_t\nabla u\cdot\nabla u dx+4\int_\Omega u_t^2 dx. \tag{2.7}$$

Lemma 2.1. *Let*

$$K_1\geq\frac{A_3}{A_1}>0 \tag{2.8}$$

be a constant and the coefficient $a(x,t)$ satisfy (1.1) and (1.2). Then there holds the inequality

$$FF''-(F')^2\geq K_1FF'. \tag{2.9}$$

Proof. Combining (2.1) with (2.2), (2.3) and (2.7), we have

$$\begin{aligned} FF'' - (F')^2 - K_1 FF' = 4\int_\Omega u^2 dx \int_\Omega u_t^2 dx - 2\int_\Omega u^2 dx \int_\Omega a_t \nabla u \cdot \nabla u dx \\ - 4\left(\int_\Omega u u_t dx\right)^2 + 2K_1 \int_\Omega u^2 dx \int_\Omega a \nabla u \cdot \nabla u dx. \end{aligned} \tag{2.10}$$

According to the Hölder's inequality, it is easy to know

$$4\int_\Omega u^2 dx \int_\Omega u_t^2 dx - 4\left(\int_\Omega u u_t dx\right)^2 \geq 0. \tag{2.11}$$

Inserting inequality (2.11) and assumption (2.8) in formula (2.10), inequality (2.9) is proved. □

Remark 2.2. *If the coefficient $a(x,t) = a(x)$ is only spacewise, then inequality (2.9) becomes particularly simple. If $a_t(x,t) = 0$, then $K_1 = 0$ and*

$$FF'' - (F')^2 \geq 0, \tag{2.12}$$

i.e., $\ln(F(t))$ *is logarithmic convex. This result can also be found in [1].* □

Now, we use the differential inequality (2.9) to derive a stability estimate for problem (1.1).

Firstly, we define the function

$$\sigma := e^{K_1 t}, \tag{2.13}$$

and then regard F as a function of σ. Let us introduce an auxiliary function

$$G(\sigma) := \log[F(t(\sigma))], \tag{2.14}$$

and a little bit of calculus shows that

$$\frac{d^2 G(\sigma)}{d\sigma^2} = \frac{FF'' - (F')^2 - K_1 FF'}{(FK_1\sigma)^2}. \tag{2.15}$$

Theorem 2.1 implies that $\frac{d^2G(\sigma)}{d\sigma^2} \geq 0$, i.e., $G(\sigma)$ is convex on the interval $1 \leq \sigma \leq \sigma_1$ with $\sigma_1 := e^{K_1 T}$.

According to the convex property of function $G(\sigma)$, we have

$$G(\sigma) \leq \frac{\sigma - 1}{\sigma_1 - 1} G(\sigma_1) + \frac{\sigma_1 - \sigma}{\sigma_1 - 1} G(1). \tag{2.16}$$

In addition, from (2.14), inequality (2.16) is equivalent to

$$F(t) \leq [F(T)]^{\frac{\sigma-1}{\sigma_1-1}} [F(0)]^{\frac{\sigma_1-\sigma}{\sigma_1-1}}. \tag{2.17}$$

For notational simplicity, we define

$$\nu(t) := \frac{\sigma - 1}{\sigma_1 - 1} = \frac{e^{K_1 t} - 1}{e^{K_1 T} - 1} > 0, \tag{2.18}$$

and inequality (2.17) can be rewritten as

$$F(t) \leq [F(T)]^{\nu(t)} [F(0)]^{1-\nu(t)}. \tag{2.19}$$

It is obvious that $0 \leq \nu(t) \leq 1$.

As for any ill-posed problems, in order to obtain continuous dependence on the data, we assume the solutions of problem (1.1) will satisfy a traditional *a-priori* condition:

$$\|u(\cdot, 0)\|_{L^2(\Omega)} = \left(\int_\Omega u^2(x, 0) dx \right)^{\frac{1}{2}} \leq E, \tag{2.20}$$

where E is a constant. Moreover, we assume that there holds

$$\|\varphi_1 - \varphi_2\|_{L^2(\Omega)} < \delta, \tag{2.21}$$

where φ_1 and φ_2 denote two different exact final data functions.

Once the existence of solution in the weak sense is assumed, then the main stability result can be established as follows.

Theorem 2.3. *Assume that the two solutions u_1 and u_2 satisfy system (1.1) with assumptions (2.20) and (2.21). Then, for $0 \leq t \leq T$, there is a stability estimate of Hölder type*

$$\|u_1(\cdot, t) - u_2(\cdot, t)\| \leq 2\delta^{\nu(t)} E^{1-\nu(t)}, \tag{2.22}$$

where $\nu(t)$ is given by (2.18).

Proof. Inserting $u_1 - u_2$ into the expression for $F(t)$ given by (2.1), and using the triangle inequality together with assumptions (2.20) and (2.21), we get

$$\begin{aligned} &\|u_1(\cdot, t) - u_2(\cdot, t)\|^2 \\ \leq &\|u_1(\cdot, 1) - u_2(\cdot, 1)\|^{2\nu(t)} \|u_1(\cdot, 0) - u_2(\cdot, 0)\|^{2(1-\nu(t))} \\ \leq &\delta^{2\nu(t)} (2E)^{2(1-\nu(t))}. \end{aligned}$$

The stability result (2.22) now is proved. □

Remark 2.4. *If the coefficient $a(x, t)$ does not depend on t, i.e., $a_t = 0$, then there holds a more simple stability estimate*

$$\|u_1(\cdot, t) - u_2(\cdot, t)\| \leq 2\delta^t E^{1-t}, \tag{2.23}$$

which is also demonstrated in [1]. □

Remark 2.5. *If we want to use the logarithmic convexity method, we should firstly find an appropriate energy function F. In Theorem 2.2 of [1], K.A. Ames and J.F. Epperson let F satisfy $FF'' - (F')^2 \geq 0$. And in papers [8, 18] that deal with another ill-posed problem, F satisfies $FF'' - (F')^2 \geq -K_1FF' - K_2F^2$. While in this paper, we let F satisfy $FF'' - (F')^2 \geq K_1FF' - K_2F^2$. Hence, it is easy to see that if we can find an appropriate energy function F satisfying*

$$FF'' - (F')^2 \geq \pm K_1FF' - K_2F^2, \tag{2.24}$$

where $K_1 \geq 0, K_2 \geq 0$ are constants, then we can easily use the logarithmic convexity method to get conditional stability. □

Corollary 2.6. *Given a function $\varphi(x) \in H_0^1(\Omega)$, there is at most one function $u(\cdot, t) \in H_0^1(\Omega)$ satisfying problem (1.1).*
Proof. It is easy to get the uniqueness of problem (1.1) from Theorem 2.3. □

Remark 2.7. *If the boundary conditions of problem (1.1) are changed into the following*

$$\begin{cases} u|_{\Gamma_1} = 0, \quad or \quad \frac{\partial u}{\partial n}|_{\Gamma_1} = 0, \quad t \in [0,T], \\ u|_{\Gamma_2} = 0, \quad or \quad \frac{\partial u}{\partial n}|_{\Gamma_2} = 0, \quad t \in [0,T], \end{cases} \tag{2.25}$$

where $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$, then the results of all the above are still right. □

3 Formal Solution and Regularization

3.1 Formal solution

In this subsection we will derive a formal solution of the BPDE in terms of the eigenfunctions of the Laplace operator. We first state a few properties of the eigenvalues of the operator $-\Delta$ on the open, connected bounded domain Ω with dirichlet boundary conditions. One can also refer to chapter 6.5 in [11].
Theorem 3.1. (Eigenvalues of the Laplace operator)
(i) Each eigenvalues of $-\Delta$ is real.
(ii) Furthermore, if we repeat each eigenvalue according to its (finite) multiplicity, we have

$$\Sigma = \{\lambda_k\}_{k=1}^{\infty},$$

where

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots,$$

and

$$\lambda_k \to \infty \qquad as \quad k \to \infty,$$

(iii) Finally, there exists an orthonormal basis $\{w_k\}_{k=1}^{\infty}$ of $L^2(\Omega)$, where $w_k \in H_0^1(U)$ is an eigenfunction corresponding to λ_k:

$$\begin{cases} -\triangle w_k = \lambda_k w_k, & in \quad \Omega, \\ w_k = 0, & on \quad \partial\Omega, \end{cases}$$

for $k = 1, 2, \dots$. □

According to Theorem 3.1, it is reasonable to look for a function $u \in H_0^1(\Omega)$ of the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t) w_n(x), \tag{3.1}$$

where w_n is the complete set of eigenfunctions for $-\triangle$ in $H_0^1(\Omega)$. Set $u_n(x,t) := c_n(t)w_n(x)$, there is $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$. If $u_n(x,t), n = 1, \dots, \infty$ satisfy the following problem

$$\begin{cases} (u_n)_t = \nabla \cdot (a(x,t)\nabla u_n), & x \in \Omega, t \in [0,T], \\ u_n(x,T) = (\varphi, w_n)w_n(x), & x \in \Omega, \end{cases} \tag{3.2}$$

then it is easy to see that $u(x,t)$ satisfies problem (1.1). Therefore, we only need to obtain $u_n(x,t)$, for $n = 1, \dots, \infty$.

Since $w_n(x) \in H_0^1(\Omega)$, then $u_n(\cdot,t) = c_n(t)w_n(\cdot) \in H_0^1(\Omega)$. For any $v \in H_0^1(\Omega)$, from (3.2) we know that $u_n(x,t)$ satisfies

$$-\int_\Omega a(x,t)\nabla u_n \cdot \nabla v dx = \int_\Omega (u_n)_t v dx, \qquad \forall v \in H_0^1(\Omega). \tag{3.3}$$

Especially, if we choose $v = w_n$, there also holds

$$-\int_\Omega a(x,t)\nabla u_n \cdot \nabla w_n dx = \int_\Omega (u_n)_t w_n dx,$$

i.e.,

$$-c_n(t)\int_\Omega a(x,t)\nabla w_n \cdot \nabla w_n dx = c_n'(t)\int_\Omega w_n w_n dx = c_n'(t), \tag{3.4}$$

from Theorem 3.1 (iii). Since $u_n(x,T) = (\varphi, w_n)w_n(x)$, $c_n(t)$ satisfies

$$c_n(T) = (\varphi, w_n), \qquad \text{for} \quad n = 1, 2, \dots. \tag{3.5}$$

From (3.4) and (3.5), it is easy to obtain $c_n(t)$ as

$$c_n(t) = (\varphi, w_n)\exp(-\int_t^T b_n(t)dt), \qquad \text{for} \quad n = 1, 2, \dots \tag{3.6}$$

where $b_n(t) := -\int_\Omega a(x,t)\nabla w_n \cdot \nabla w_n dx$. Therefore, according to the uniqueness of the solution (see Corollary 2.6), there hold

$$u(x,t) = \sum_{n=1}^{\infty} \exp[\int_t^T (\int_\Omega a(x,t)\nabla w_n \cdot \nabla w_n dx)dt](\varphi, w_n)w_n(x), \tag{3.7}$$

$$\psi(x) = \sum_{n=1}^{\infty} \exp[\int_0^T (\int_\Omega a(x,t)\nabla w_n \cdot \nabla w_n dx)dt](\varphi, w_n)w_n(x). \tag{3.8}$$

These are attractive representation formulas for the solution, but depend upon our being able to find eigenfunctions and constants satisfying (3.5).

3.2 Ill-posedness

In order to show the ill-posedness of the BPDE clearly, we would like to list some classical ill-posed problems here. Furthermore, to compare the ill-posedness, we try to give their solutions with similar form to the BPDE.

(a) High order numerical derivatives

$$\begin{aligned} &f(x) \in L^2(-\pi,\pi), \\ &f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx, \\ &f^{(k)}(x) = \sum_{n=-\infty}^{\infty} (in)^k c_n e^{inx}. \end{aligned}$$

(b) An inverse spacewise-dependent heat source problem [23]

$$\begin{cases} u_t = \Delta u + f(x), & x \in \Omega, t \in [0,T], \\ u(x,0) = 0, & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, t \in [0,T] \\ u(x,t_0) = g(x), & x \in \Omega, t_0 \in (0,T], \end{cases}$$

solve the source term $f(x)$

$$f(x) = \sum_{n=1}^{\infty} \frac{\lambda_n}{1 - e^{-\lambda_n t_0}}(g, w_n)w_n.$$

(c) A Cauchy problem of an elliptic equation on a cylindrical domain [9]

$$\begin{cases} u_{xx} + \Delta_y u = 0, & x \in [0,1], y \in \Omega, \\ u(x,y) = 0, & y \in \Omega, \\ u(0,y) = g(y), & y \in \Omega, \\ u_x(0,y) = 0, & y \in \Omega, \end{cases}$$

solve $u(1,y) = f(y)$

$$\begin{aligned} u(x,y) &= \sum_{n=1}^{\infty} \cosh(\sqrt{\lambda_n}x)(g, w_n)w_n, \\ f(y) &= \sum_{n=1}^{\infty} \cosh(\sqrt{\lambda_n})(g, w_n)w_n. \end{aligned}$$

(d) An inverse heat conduction problem

$$\begin{cases} u_t = u_{xx} + \Delta_y u = 0, & x \in [0,1], y \in \Omega, t \in (0,\infty), \\ u(x,y,0) = 0, & x \in [0,1], y \in \Omega, \\ u(x,y,t) = 0, & x \in [0,1], y \in \partial\Omega, t \in (0,\infty), \\ u(0,y,t) = g(y,t), & y \in \Omega, t \in (0,\infty), \\ u_x(0,y,t) = 0, & y \in \Omega, t \in (0,\infty), \end{cases}$$

solve $u(1,y,t) = f(y,t)$

$$\hat{u}(x,y,\xi) = \sum_{n=1}^{\infty} \cosh(x\theta_1)(\hat{g}(\cdot,\xi), w_n)w_n,$$

where

$$\theta_1 = \sqrt{i\xi + \lambda_n} = \sqrt{\frac{\sqrt{\xi^2+\lambda_n^2}+\lambda_n}{2}} + isign(\xi)\sqrt{\frac{\sqrt{\xi^2+\lambda_n^2}-\lambda_n}{2}}$$

(e) A backward heat conduction problem (BHCP) [5]

$$\begin{cases} u_t = a\Delta u, & x \in \Omega, t \in [0,T], \\ u(x,T) = g(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, t \in [0,T], \end{cases}$$

solve $u(x,t)$ for $0 \le t < T$

$$u(x,t) = \sum_{n=1}^{\infty} e^{a\lambda_n(T-t)}(g, w_n)w_n.$$

We list the 'kernel's of these classical ill-posed problems: (a) $(in)^k$, (b) $\frac{\lambda_n}{1-e^{-\lambda_n t_0}}$, (c) $\cosh(\sqrt{\lambda_n})$, (d) $\cosh(x\theta_1)$, (e) $e^{a\lambda_n(T-t)}$, where the 'kernel' can be understood as [20]. While the BPDE has the 'kernel' as

$$k(t) = \exp(\int_t^T (\int_\Omega a(x,t)\nabla w_n \cdot \nabla w_n dx)dt). \tag{3.9}$$

According to the Lebesgue mean-value theorem of integrals, there exists some point $x_n(t) \in \Omega$ for every t, so that

$$k(t) = \exp(\int_t^T (a(x_n(t),t) \int_\Omega \nabla w_n \cdot \nabla w_n dx)dt). \tag{3.10}$$

Based on Theorem 3.1 (iii), there holds

$$\int_\Omega \nabla w_n \cdot \nabla w_n dx = -\int_\Omega w_n \Delta w_n dx = \int_\Omega w_n \lambda_n w_n dx = \lambda_n,$$

therefore (3.10) becomes

$$k(t) = \exp(\lambda_n \int_t^T a(x_n(t), t)dt). \tag{3.11}$$

Through observing (3.11), the BPDE has a similar 'kernel' as the BHCP, qualitatively the BPDE and BHCP are about as severely ill-posed and are the most ill-posed among the above six inverse problems.

3.3 A Regularization Method

Since $0 < A_1 \leq a(x,t)$, $\exp(\lambda_n \int_t^T a(x_n(t), t)dt)$ increases exponentially fast, which shows that problem (1.1) is severely ill-posed. Moreover, if $A_1 \geq 1$, it is even more ill-posed than BHCP [12]. Therefore, its numerical treatment requires some special regularization methods. A simple idea is to eliminate all large eigenvalues from the solution, and instead consider (3.7) with noisy data $\varphi^\delta(x)$ only for $n \leq N$, where N is an undetermined value, i.e., we construct a regularized solution as

$$u^{N,\delta}(x,t) := \sum_{n=1}^{N} \exp[\int_t^T (\int_\Omega a(x,t)\nabla w_n \cdot \nabla w_n dx)dt](\varphi^\delta, w_n)w_n(x). \tag{3.12}$$

The convergence with respect to δ can be arbitrary slow if no *a-priori* assumption about the solution [10]. Before considering the error estimate between the exact and regularized solutions, the noise level of the noisy data and some *a-priori* bound for the exact solution should be given:

$$\|\varphi^\delta - \varphi\|_{L^2(\Omega)} \leq \delta, \tag{3.13}$$

$$\|\psi\|_{L^2(\Omega)} \leq E. \tag{3.14}$$

Then the error estimate satisfies

$$\|u - u^{N,\delta}\| \leq \|u - u^N\| + \|u^N - u^{N,\delta}\|. \tag{3.15}$$

For the second term in the right hand side of (3.15), we can estimate as

$$\begin{aligned}
&\|u^N - u^{N,\delta}\| \\
=&\|\sum_{n=1}^{N} \exp[-\int_t^T b_n(t)dt](\varphi, w_n)w_n - \sum_{n=1}^{N} \exp[-\int_t^T b_n(t)dt](\varphi^\delta, w_n)w_n\| \\
=&\|\sum_{n=1}^{N} \exp[-\int_t^T b_n(t)dt](\varphi - \varphi^\delta, w_n)w_n\| \\
\leq& \max_{1\leq n\leq N} \exp[-\int_t^T b_n(t)dt](\sum_{n=1}^{N} |(\varphi - \varphi^\delta, w_n)|^2)^{1/2} \\
\leq& \max_{1\leq n\leq N} \exp[-\int_t^T (-\int_\Omega a(x,t)\nabla w_n \cdot \nabla w_n dx)dt](\sum_{n=1}^{\infty} |(\varphi - \varphi^\delta, w_n)|^2)^{1/2}.
\end{aligned}$$

According to the Lebesgue mean-value theorem of integrals, the above inequality becomes

$$\begin{aligned}\|u^N - u^{N,\delta}\| &\leq \max_{1\leq n\leq N} \exp[\int_t^T a_n(x(t),t)(\int_\Omega \nabla w_n \cdot \nabla w_n dx)dt]\delta \\ &= \max_{1\leq n\leq N} \exp[\int_t^T a_n(x(t),t)\lambda_n dt]\delta \\ &\leq \max_{1\leq n\leq N} \exp[A_2(T-t)\lambda_n]\delta \\ &\leq \exp[A_2(T-t)\lambda_N]\delta, \end{aligned} \tag{3.16}$$

where A_2 is the constant in (1.2).

On the other hand, there holds

$$\begin{aligned} &\|u-u^N\| \\ =&\|\sum_{n=1}^{\infty}\exp[-\int_t^T b_n(t)dt](\varphi,w_n)w_n - \sum_{n=1}^{N}\exp[-\int_t^T b_n(t)dt](\varphi,w_n)w_n\| \\ =&\|\sum_{n=N+1}^{\infty}\exp[-\int_t^T b_n(t)dt](\varphi,w_n)w_n\| \\ =&\|\sum_{n=N+1}^{\infty}\exp[-\int_t^T(-\int_\Omega a(x,t)\nabla w_n\cdot\nabla w_n dx)dt](\varphi,w_n)w_n\| \\ =&\|\sum_{n=N+1}^{\infty}\exp[-\int_0^t(\int_\Omega a(x,t)\nabla w_n\cdot\nabla w_n dx)dt]\exp[\int_0^T(\int_\Omega a(x,t)\nabla w_n\cdot\nabla w_n dx)dt](\varphi,w_n)w_n\| \\ \leq&\|\sum_{n=N+1}^{\infty}\exp[-\int_0^t(\int_\Omega a(x,t)\nabla w_n\cdot\nabla w_n dx)dt](u(x,0),w_n)w_n\|. \end{aligned}$$

Based on the Lebesgue mean-value theorem of integrals, there holds

$$\begin{aligned}\|u-u^N\| &\leq \|\sum_{n=N+1}^{\infty}\exp[-\int_0^t a_n(x(t),t)(\int_\Omega \nabla w_n\cdot\nabla w_n dx)dt](u(x,0),w_n)w_n\| \\ &= \|\sum_{n=N+1}^{\infty}\exp[-\int_0^t a_n(x(t),t)\lambda_n dt](u(x,0),w_n)w_n\| \\ &\leq \max_{n\geq N+1}\exp[-\int_0^t a_n(x(t),t)\lambda_n dt]\|\sum_{n=N+1}^{\infty}(u(x,0),w_n)w_n\| \\ &\leq \max_{n\geq N+1}\exp[-A_1\lambda_n t]\|u(\cdot,0)\| \\ &\leq \exp[-A_1\lambda_{N+1}t]\|u(\cdot,0)\|, \end{aligned} \tag{3.17}$$

where A_1 is the same as in (1.2).

According to (3.15)–(3.17), the following result holds.

Theorem 3.2. *Let $u^{N,\delta}(x,t)$ be the approximation of the solution $u(x,t)$ of problem (1.1). If conditions (3.13) and (3.14) hold, then there holds the error estimate:*

$$\| u^{N,\delta} - u \|_{L^2(\Omega)} \leq \exp[A_2(T-t)\lambda_N]\delta + \exp[-A_1\lambda_{N+1}t]E. \tag{3.18}$$

Remark 3.3. *Theorem 3.2 only gives the error estimate in the interior of the interval $[0,T)$. In order to obtain the error estimate for the case $t=0$, the stronger* a-priori *bound for*

$$\|\psi\|_{H^p(\Omega)} := \left(\int_\Omega (\sum_{|\alpha|\leq p} D^\alpha \psi \overline{D^\alpha \psi}) dx \right)^{\frac{1}{2}} \leq E$$

must be imposed (see e.g. Theorem 3.2 in [20]). In the following we consider the case $p=1$, i.e.,

$$\|\psi\|_{H^1(\Omega)} := \left(\int_\Omega (\sum_{|\alpha|\leq 1} D^\alpha \psi \overline{D^\alpha \psi}) dx \right)^{\frac{1}{2}} \leq E. \tag{3.19}$$

Firstly, we can obtain the following result about the norm in $H^1(\Omega)$.

Lemma 3.4. *Assume that w_n, λ_n are the same as in Theorem 3.1. If $v(x) \in H_0^1(\Omega)$, then*

$$\|v\|^2_{H^1(\Omega)} = \sum_{n=1}^{\infty} (1+\lambda_n)|d_n|^2, \tag{3.20}$$

where $d_n = (v, w_n)$.

Proof. On one hand, because $v(x) \in H_0^1(\Omega)$, according to the definition of the norm (3.19) and the Green's first identity, there is

$$\begin{aligned} \|v\|^2_{H^1(\Omega)} &= \int_\Omega (v\overline{v})dx + \int_\Omega (Dv \cdot \overline{Dv})dx, \\ &= \int_\Omega (v\overline{v})dx - \int_\Omega (\triangle v \cdot \overline{v})dx. \end{aligned} \tag{3.21}$$

On the other hand, from Theorem 3.1, it is easy to see that

$$v(x) = \sum_{n=1}^{\infty} d_n w_n(x). \tag{3.22}$$

Hence, there is

$$\|v\|^2_{H^1(\Omega)} = \sum_{n=1}^{\infty} d_n \overline{d_n} + \sum_{n=1}^{\infty} \lambda_n d_n \overline{d_n} = \sum_{n=1}^{\infty} (1+\lambda_n)|d_n|^2.$$

□

Now the relation between the exact and regularized solutions can be given by the following.

Theorem 3.5. *Let $\psi^{N,\delta}(x)$ be the approximation of the solution $\psi(x)$ of problem (1.1) at $t = 0$. If conditions (3.13) and (3.19) hold, then there holds the error estimate:*

$$\| \psi^{N,\delta} - \psi \|_{L^2(\Omega)} \leq \exp(\lambda_N A_2 T)\delta + \lambda_N^{-\frac{1}{2}} E. \tag{3.23}$$

Proof. Using the triangle inequality, we have

$$\| \psi^{N,\delta} - \psi \|_{L^2(\Omega)} \leq \| \psi^{N,\delta} - \psi^N \|_{L^2(\Omega)} + \| \psi^N - \psi \|_{L^2(\Omega)} . \tag{3.24}$$

For the first term of the right hand side above, similar to (3.16), there exists

$$\| \psi^{N,\delta} - \psi^N \|_{L^2(\Omega)} \leq \exp(\lambda_N A_2 T)\delta. \tag{3.25}$$

For the second term of the right hand side in (3.24), from condition (3.19) and Lemma 3.4, there holds

$$\begin{aligned}
\| \psi^N - \psi \|_{L^2(\Omega)} &= \| \sum_{n=N+1}^{\infty} (\varphi, w_n) \exp[\int_0^T (\int_\Omega a(x,t)\nabla w_n \cdot \nabla w_n dx)dt] w_n \| \\
&= \| \sum_{n=N+1}^{\infty} (\varphi, w_n) \exp[\lambda_n \int_0^T a(x_n,t)dt] w_n \| \\
&= \| \sum_{n=N+1}^{\infty} (\varphi, w_n) \exp[\lambda_n \int_0^T a(x_n,t)dt] (1+\lambda_n)^{\frac{1}{2}} (1+\lambda_n)^{-\frac{1}{2}} w_n \| \\
&\leq \sup_{n \geq N+1} (1+\lambda_n)^{-\frac{1}{2}} E \leq \lambda_N^{-\frac{1}{2}} E. \qquad (3.26)
\end{aligned}$$

Inserting (3.25) and (3.26) into (3.24), the error estimate (3.23) is obtained. □

The choice of a suitable value of the regularization parameter is crucial for the accuracy of the final numerical solution. Here, we provide an appropriate law to the *a-priori* choices of the regularization parameter N.

Remark 3.6. *From Theorems 3.2 and 3.5, it is easy to see that if we choose some $N(\delta)$ which satisfies the following two conditions:*

(1). $N(\delta) \to \infty$, as $\delta \to 0$;

(2). $\exp(\lambda_N A_2 T)\delta \to 0$, as $\delta \to 0$,

then $\| u^{N,\delta} - u \|_{L^2(\Omega)} \to 0$, for $t \in [0, T)$. □

Remark 3.7. *From not only the expression of the exact solution but also the* a-priori *bound provided above, we can see that the ill-posedness for the case $t = 0$ is stronger than the case $t \in (0, T)$, therefore we are very interested in the case*

$t = 0$ and will only consider it later.

To further demonstrate how to apply our regularization method to choose the regularization parameter, we give the following example in a simple case $\Omega_1 = (0, \pi) \times \cdots \times (0, \pi) \subset \mathbb{R}^m$.
Example

$$\begin{cases} u_t = \nabla \cdot (a(x,t)\nabla u), & x \in \Omega_1, 0 < t < T, \\ u(x,T) = \varphi^\delta(x), & x \in \Omega_1, \\ u(x,t) = 0, & x \in \partial\Omega_1, 0 \le t \le T. \end{cases} \tag{3.27}$$

Here our task is to approximate the initial data $u(x, 0) = \psi(x)$. The eigenvalues of $-\Delta$ are $\lambda_n = |n|^2 = \sum_{k=1}^m n_k^2$ and the corresponding eigenfunctions are $w_n = \Pi_{k=1}^m \sqrt{\frac{2}{\pi}} \sin(n_k x)$, where $n = (n_1, \cdots, n_m)$[4].

Applying Theorem 3.1, error estimate (3.24) becomes

$$\| \psi^{N,\delta} - \psi \|_{L^2(\Omega_1)} \le \exp(|N|^2 A_2 T)\delta + \frac{E}{|N|} \text{ with } N = (N_1, \cdots, N_m). \tag{3.28}$$

Then in view of Remark 3.6, we deduce
Remark 3.8. *If we choose the regularization parameter N as*

$$N_k(\delta) = \left\lfloor \left(\frac{\ln(\frac{1}{\delta})}{2A_2 T} \right)^{\frac{1}{2}} \right\rfloor, \quad k = 1, \cdots, m, \tag{3.29}$$

then error estimate (3.28) converges at 0 as $\delta \to 0$. Here $\lfloor e \rfloor$ represents the largest integer no more than e. □

Remark 3.9. *The regularization method depends on the knowledge of the eigenfunctions of the Laplace equation on the space domain. So if they are known, the method can be implemented.* □

4 Numerical implementation

In the discussion of the numerical implementation we will assume that the approximate solution (3.12) is to be computed for $t = 0$. In the cases when the eigenfunctions of the Laplacian are known explicitly (essentially for rectangular geometries) it is straightforward to implement the method numerically. Then the main computation is to compute N integrals over Ω, which may even be done analytically. We give such examples below, in Section 5.

In the case of a general geometry, the first problem is to compute approximations of the eigenfunctions corresponding to the N smallest eigenvalues. Note that for a realistic noise level (in thermal engineering applications 0.1-1 % is common) and time interval, the number of eigenfunctions required can be quite small, of the order 10, say. Thus approximations of the eigenfunctions can easily be computed from a discretization of the Laplace equation over Ω using standard software [3] (e.g. implemented in the function `eigs` in MATLAB).

Given approximations $\hat{w}_n$ of the eigenfunctions, we then need to compute the integrals (3.6)

$$\int_0^T b_n(t)dt = -\int_0^T \left(\int_\Omega a(x,t)\nabla \hat{w}_n \cdot \nabla \hat{w}_n dx \right) dt, \qquad n = 1, 2, \ldots, N.$$

This can be done by numerical quadrature,

$$\int_0^T b_n(t)dt \approx \sum_{i=1}^q \alpha_i b_n(t_i), \tag{4.1}$$

where the nodes t_i are in the interval $[0, T]$, and the weights α_i are specified by the quadrature rule. The core problem is then to compute

$$b_n(t_i) = -\int_\Omega a(x,t_i)\nabla \hat{w}_n \cdot \nabla \hat{w}_n dx, \qquad i = 1, \ldots, q, \quad n = 1, \ldots, N. \tag{4.2}$$

It turns out that these integrals are strongly related to those in the variational form of the elliptic problems

$$\nabla \cdot (a(x,t_i)\nabla u) = f, \qquad \text{in} \quad \Omega, \qquad i = 1, \ldots, q, \tag{4.3}$$

see e.g. [13, Chapter 2]. Therefore, by discretizing the elliptic equations (4.3), for instance by the finite element method, giving stiffness matrices K_i, and assuming that in that discretization the eigenfunctions are represented by vectors W_n, the integrals are approximated by

$$b_n(t_i) \approx W_n^T K_i W_n, \qquad i = 1, \ldots, q, \quad n = 1, \ldots, N.$$

We summarize the computations:

1. Discretize the Laplace equation over Ω giving a stiffness matrix L, and compute the first eigenvectors $W_n, \quad n = 1, 2, \ldots, N$.
2. Compute the stiffness matrices $K_i, i = 1, \ldots, q$, and evaluate

$$\int_0^T b_n(t)dt \approx \sum_{i=1}^q \alpha_i W_n^T K_i W_n, \qquad n = 1, 2, \ldots, N.$$

Note that it is preferable to order the computations in such a way that we only need store one stiffness matrix at a time.

5 Numerical experiments

It is difficult to find an example with an exact solution of problem (1.1), so it is necessary to firstly compute the following direct problem

$$\begin{cases} u_t = \nabla \cdot (a(x,t)\nabla u), & x \in \Omega, t \in [0,T], \\ u(x,0) = \psi(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, t \in [0,T] \end{cases} \tag{5.1}$$

to get the final data $\varphi(x)$ and constitute some examples.

To solve direct problem (5.1) numerically, one can discretize it only in space and give a system of ODE's

$$U_t = A(t)U \tag{5.2}$$

where $U_i(t) = u(x_i, t)$ and $U_i(0) = \psi(x_i)$.

The system can then be solved using a standard ODE solver, e.g. ODE23s in Matlab.

For testing the proposed regularization method, we use the following four examples.

Example 1. We consider

$$\begin{cases} u_t = ((xt+1)u_x)_x, & 0 < x < \pi, 0 < t < T, \\ u(x,0) = \psi(x), & 0 \le x \le \pi, \\ u(0,t) = u(\pi,t) = 0, & 0 \le t \le T, \end{cases} \tag{5.3}$$

with a smooth function

$$\psi(x) = \sin(x). \tag{5.4}$$

Example 2. In example 1, we choose the solution as the first eigenvector of the Laplace operator, which should make the problem quite easy. Therefore, here we come up with a more general solution that is spanned by some more eigenvectors as follows

$$\psi(x) = \sin(x) + \sin^2(2x). \tag{5.5}$$

Example 3. We consider the same problem as (5.3) with a discontinuous function

$$\psi(x) = \begin{cases} 0, & 0 \le x < \frac{\pi}{4}, \\ -1, & \frac{\pi}{4} \le x < \frac{\pi}{2}, \\ 1, & \frac{\pi}{2} \le x \le \frac{3\pi}{4}, \\ 0, & \frac{3\pi}{4} < x \le \pi. \end{cases} \tag{5.6}$$

Example 4. We give the following problem in a simple two-dimensional case $\Omega_1 = (0,\pi) \times (0,\pi)$.

$$\begin{cases} u_t = (a(x,y,t)u_x)_x + (a(x,y,t)u_y)_y, & (x,y) \in \Omega_1, 0 < t < T, \\ u(x,y,0) = \psi(x,y), & (x,y) \in \Omega_1, \\ u(x,y,t) = 0, & (x,y) \in \partial\Omega_1, 0 \le t \le T. \end{cases} \tag{5.7}$$

Here the coefficient and the exact initial data are defined by

$$a(x,y,t) = xyt + 1, \qquad \psi(x,y) = x\sin(x)\sin(y). \tag{5.8}$$

To check the computational feasibility of the proposed procedure, we perform some numerical experiments. All tests are performed using *Matlab 7.0.*

And we use Simpson's rule to approximate the integrals.

Test 1. Consider the one-dimensional examples 1–3.

To compute the direct problem, it is important to get the ODE's system (5.2) first. Using the finite difference

$$\left[a(x_{i+\frac{1}{2}},t)\frac{u(x_{i+1},t)-u(x_i,t)}{h}-a(x_{i-\frac{1}{2}},t)\frac{u(x_i,t)-u(x_{i-1},t)}{h}\right]/h \qquad (5.9)$$

to approximate $(au_x)_x(x_i,t)$, we discretize problem (5.3) as (5.2).

In our numerical experiments of inverse problem, the key is to know $w_n(t)$. Fortunately, the eigenfunctions are $w_n=\sqrt{\frac{2}{\pi}}\sin(nx)$ for problem (4.5).

Example 1.

Figure 1 shows the computed final data $u(x,T)$ for $T=0.1,0.2,0.5,1$ by computing the direct problem of Example 1. From Figure 1 we can see that the bigger the T is, the smaller the data of $u(x,T)$ is, i.e., the less information of final data is, therefore, the more difficult to compute the inverse problem is.

Figure 2 gives the comparison of the exact solution with its computed approximations for different final values, where random noise $\epsilon=10^{-2}$ are added to the final data and the regularization parameters are chosen by $N=4,1,1$ for $T=0.1,0.5,1$, respectively. From Figure 2 we see that the computed result is acceptable. Moreover, the smaller the T is, the better the computed approximation is. Which accords with the analysis of ill-posedness degree before. The numerical results with various levels of noise in the data for $T=0.2$ are shown in Figure 3. The regularization parameters are chosen by $N=1,2,3$ for the cases of $\epsilon=10^{-1},10^{-2},10^{-3}$, respectively. From Figure 3 we can see that the approximated solutions converge towards the exact solution as the amount of noise decreases.

Example 2.

Figure 4 shows the computed final data $u(x,0.1)$, while Figure 5 gives the numerical results with different levels of noise for $T=0.1$. The regularization parameters are chosen by $N=5,6,7$ for the cases of $\epsilon=10^{-1},10^{-2},10^{-3}$, respectively. From Figure 5, we note that the numerical results are all well for different levels of noise, which indicates that the proposed method is not very sensitive to the noise.

Example 3.

It is more difficult to reconstruct the solution for the discontinuous case. The numerical approximations of $u(x,0)$ from the computed data $u(x,0.1)$ (see Figure 6) with various levels of noise in the data are shown in Figure 7. The regularization parameters are chosen by $N=4,5,7$ for $\epsilon=10^{-1},10^{-2},10^{-3}$, respectively. It can be observed from this figure that the numerical results are in good agreement with the exact solution, taking into account the ill-posed nature of the problem and discontinuity of the solution. However it fails to capture

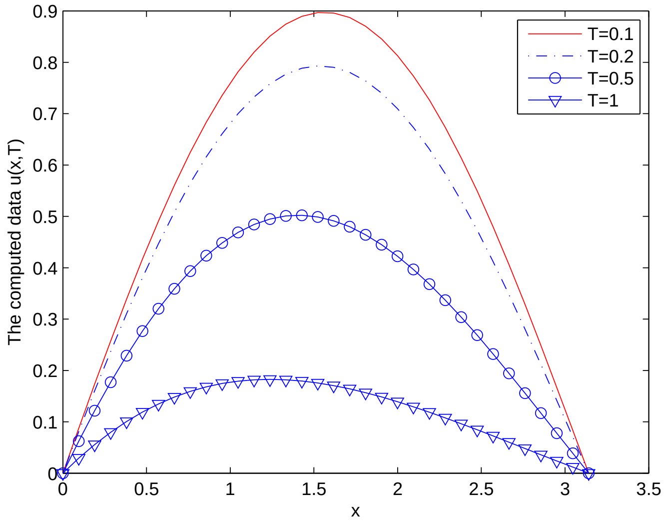

Figure 1: Example 1. The computed final data of different times $T = 0.1$ (solid line), $T = 0.2$ (dashed-dotted), $T = 0.5$ (circle) and $T = 1$ (Downward-pointing Triangle).

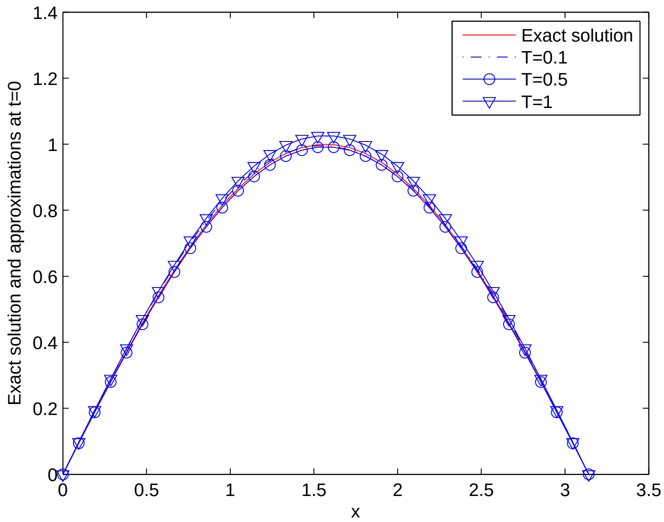

Figure 2: Example 1. The exact solution $u(x, 0)$ (solid line) and its approximations of different times $T = 0.1$ (dashed-dotted), $T = 0.5$ (circle) and $T = 1$ (Downward-pointing Triangle) with $\epsilon = 10^{-2}$.

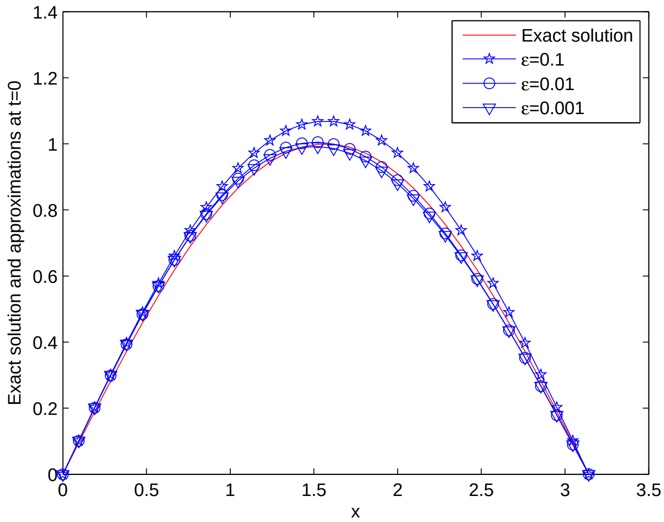

Figure 3: Example 1. The exact solution $u(x, 0)$ (solid line) and its approximations of $T = 0.2$ for various levels of noise $\epsilon = 0.1$ (pentagram), $\epsilon = 0.01$ (circle) and $\epsilon = 0.001$ (Downward-pointing Triangle).

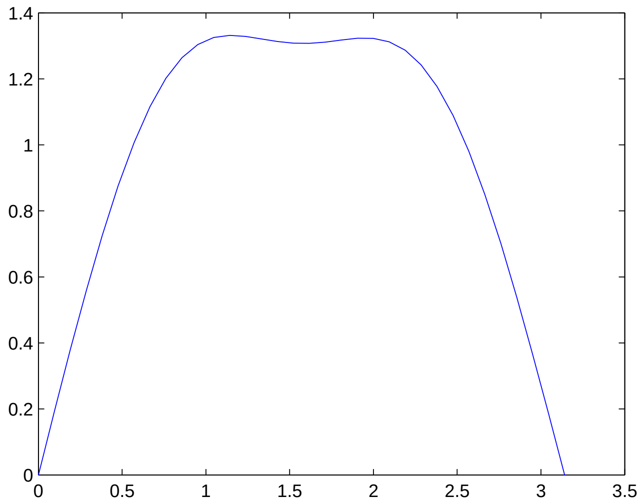

Figure 4: Example 2. The computed final data $u(x, 0.1)$.

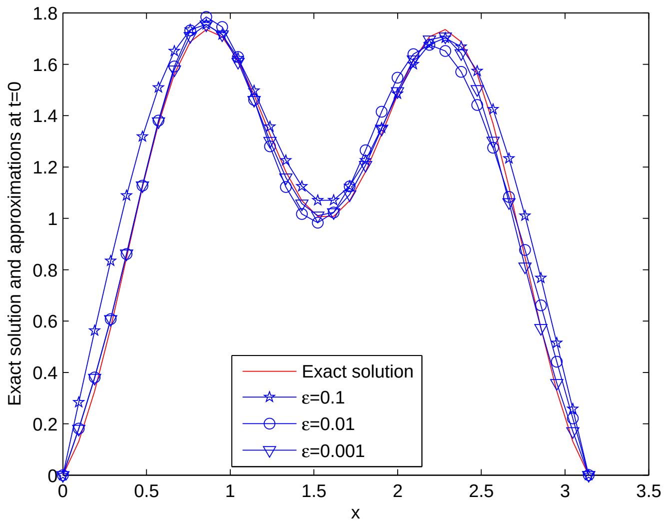

Figure 5: Example 2. The exact solution $u(x, 0)$ (solid line) and its approximations of $T = 0.1$ for various levels of noise $\epsilon = 0.1$ (pentagram), $\epsilon = 0.01$ (circle) and $\epsilon = 0.001$ (Downward-pointing Triangle).

distinct features of discontinuous function, e.g., the discontinuous points. Note that the same situation happened for the other ill-posed problems, such as the Backward Heat Conduction Problem [12].

Test 2. Consider the two-dimensional example 4.

Similar to Test 1, we discretize the direct problem by the following finite-difference scheme in space

$$\frac{1}{h^2}[a(x_{i+\frac{1}{2}}, y_j, t)u(x_{i+1}, y_j, t) + a(x_i, y_{j+\frac{1}{2}}, t)u(x_i, y_{j+1}, t) - (a(x_{i+\frac{1}{2}}, y_j, t) + a(x_{i-\frac{1}{2}}, y_j, t) + a(x_i, y_{j+\frac{1}{2}}, t) + a(x_i, y_{j-\frac{1}{2}}, t))u(x_i, y_j, t) + a(x_{i-\frac{1}{2}}, y_j, t) u(x_{i-1}, y_j, t) + a(x_i, y_{j-\frac{1}{2}}, t)u(x_i, y_{j-1}, t)] \tag{5.10}$$

to approximate $((au_x)_x + (au_y)_y)(x_i, y_j, t)$. The eigenvectors can be also easily known as $w_n(x, y) = \sqrt{\frac{2}{\pi}} \sin(n_1 x)\sqrt{\frac{2}{\pi}} \sin(n_2 y)$ [4].

Example 4.

Figure 8 presents the computed final data $u(x, y, 0.1)$. The corresponding error distributions for different regularization parameters are illustrated in Figure 9, where the noise is $\epsilon = 0.01$. From Figure 9, it can be seen that the proposed

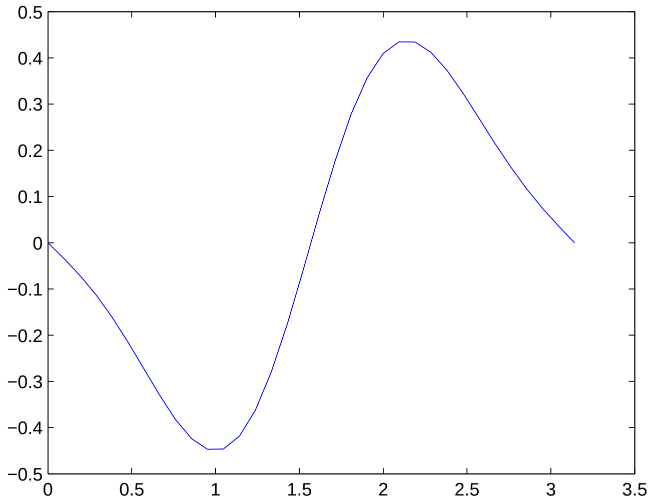

Figure 6: Example 3. The computed final data $u(x, 0.1)$.

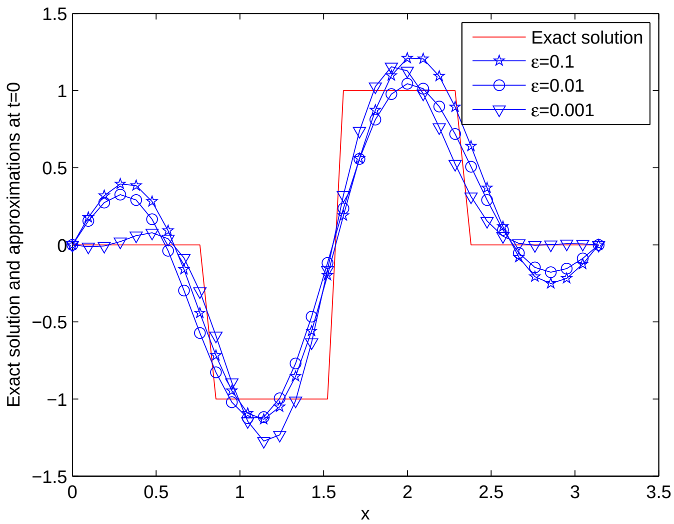

Figure 7: Example 3. The exact solution $u(x, 0)$ (solid line) and its approximations of $T = 0.1$ for various levels of noise $\epsilon = 0.1$ (pentagram), $\epsilon = 0.01$ (circle) and $\epsilon = 0.001$ (Downward-pointing Triangle).

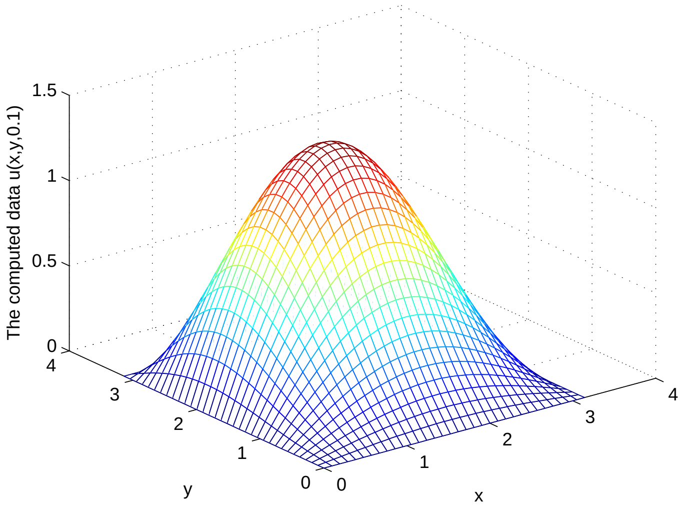

Figure 8: Example 4. The computed final data at $T = 0.1$.

method is not very sensitive to the regularization parameter. We also find that the numerical result is not better than the one-dimensional case, since it is more ill-posed for the two-dimensional case than that of the one-dimensional case.

6 Conclusion

In this paper, we considered a backward parabolic PDE with variable coefficients, and got four main results: (a) Under the assumption that there exists solutions, the uniqueness of the solution exists, even more, a stability estimate was also obtained; (b) Compared with some other ill-posed problems, the BPDE is exponentially ill-posed, and the degree of the ill-posedness is the strongest; (c) Compared with the BHCP, the BPDE has almost the same ill-posed degree, however the BPDE is more difficult to analyze in both the theoretical and numerical aspects; (d) A simple regularization method was proposed to deal with the BPDE. In addition, a rigorous criterion for the choice of the regularization parameter was given. The error estimate between the exact solution and the approximated solution and the numerical implement for general domain were proved.

We finally remark that if the coefficient $a(x, t)$ has a substantial variation with respect to x, then one may replace the Laplace operator in the overall description of the method by the elliptic operator $\nabla(\bar{a}(x)\nabla u)$, where $\bar{a}(x)$ is a mean value of $a(x, t)$ over $[0, T]$.

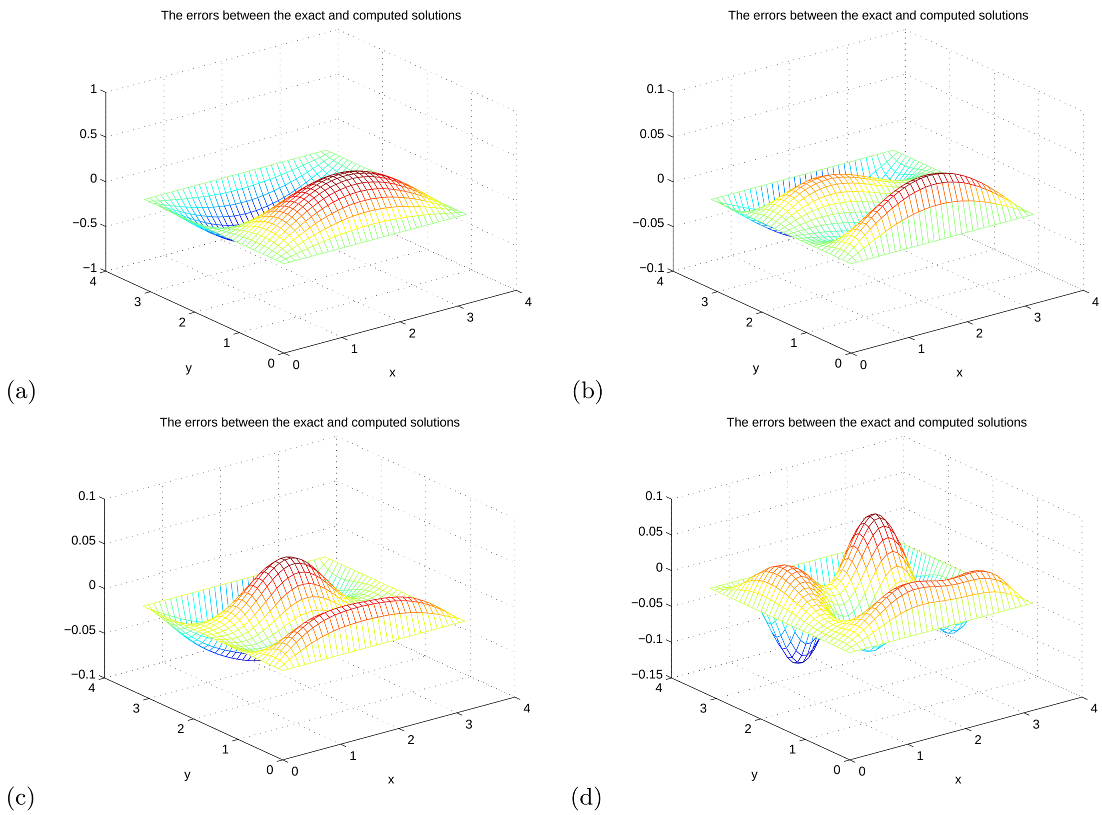

Figure 9: Example 4. The error distributions corresponding to the different regularization parameters (a) $N = (1, 1)$, (b) $N = (2, 2)$, (c) $N = (3, 3)$, (d) $N = (4, 4)$ of $T = 0.1$ and $\epsilon = 0.01$.

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References

[1] K. A. Ames and J. F. Epperson, A kernel-based method for the approximate solution of backward parabolic problems, SIAM Journal on Numerical Analysis, 34 (1997) 1357–1390.

[2] K. A. Ames, G. W. Clark, J. F. Epperson and S. F. Oppenheimer, A comparison of regularizations for an ill-posed problem, Mathematics of Computation, 67 (1998) 1451–1471.

[3] Z. J. Bai, J. Demmel, J. Dongarra, A. Ruhe and H. van der Vorst, Templates for the solution of algebraic eigenvalue problems: A practical guide, Society for Industrial and Applied Mathematics, 2000.

[4] G. Cain and G. H. Meyer, Separation of Variables for Partial Differential Equations: an eigenfunction approach, Chapman & Hall/CRC, 2006.

[5] J. Cheng and J. J. Liu, A quasi Tikhonov regularization for a two-dimensional backward heat problem by a fundamental solution, Inverse Problems, 24 (2008) 065012(18pp).

[6] J. Cheng and M. Yamamoto, One new strategy for *a priori* choice of regularizing parameters in Tikhonov's regularization, Inverse Problems, 16 (2000) L31–L38.

[7] L. Eldén, Solving the sideways heat equation by a method of lines. J. Heat Transfer, Trans. ASME, 119:406–412, 1997.

[8] L. Eldén and F. Berntsson, A stability estimate for a Cauchy problem for an elliptic partial differential equation, Inverse Problems, 21 (2005) 1643–1653.

[9] L. Eldén and V. Simoncini, Solving Ill-Posed Cauchy Problems by a Krylov Subspace Method, Inverse Problems, 25 (2009) 065002 (22pp).

[10] H. W. Engl, M. Hanke and A. Neubauer, Regularization of inverse problems, Kluwer Academic Publishers Group, Dordrecht, 1996.

[11] L. C. Evans, Partial differential equations, American Mathematical Society Providence, 1998.

[12] X. L. Feng, Z. Qian and C. L. Fu, Numerical approximation of solution of nonhomogeneous backward heat conduction problem in bounded region, Mathematics and Computers in Simulation, 79 (2008) 177–188.

[13] M. S. Gockenbach, Understanding and implementing the finite element method, Society for Industrial and Applied Mathematics, 2006.

[14] O. Y. Imanuvilov and M. Yamamoto, Lipschitz stability in inverse parabolic roblems by the Carleman estimate, Inverse Problems, 14 (1998) 1229–1245.

[15] J. J. Liu, Numerical solution of forward and backward problem for 2-D heat conduction problem, Journal of Computational and Applied Mathematics, 145 (2002) 459–482.

[16] J. M. Marbán and C. Palencia, A new numerical method for backward parabolic problems in the maximum-norm setting, SIAM Journal on Numerical Analysis, 40 (2002) 1405–1420.

[17] W. L. Miranker, A well posed problem for the backward heat equation, Proceedings of the American Mathematical Society, 12 (1961) 243–247.

[18] L. E. Payne, On a priori bounds in the Cauchy problem for elliptic equations, SIAM Journal on mathematical analysis, 1 (1970) 82–89.

[19] L. E. Payne, Improperly posed problems in partial differential equations, Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, PA, 1975.

[20] Z. Qian and C. L. Fu, Regulariztion strategies for a two-dimensional inverse heat conduction problem, Inverse Problems, 23 (2007), 1053–1068.

[21] Z. Qian, C. L. Fu and X. L. Feng, A modified method for high order numerical derivatives, Applied Mathematics and Computation, 182 (2006) 1191–1200.

[22] A. Shidfar, J. Damirchi and P. Reihani, A stable numerical algorithm for identifying the solution of an inverse problem, Applied Mathematics and Computation, 190 (2007) 231–236.

[23] L. Yan, F. L. Yang and C. L. Fu, A meshless method for solving an inverse spacewise-dependent heat source problem, Journal of Computational Physics, 228 (2009) 123–136.

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