Uniform asymptotic formulae for Green's tensors in elastic singularly perturbed domains with multiple inclusions

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In memory of Gaetano Fichera

Abstract

We present asymptotic approximations of Green's kernels for operators of linear elasticity in planar and three-dimensional domains containing multiple inclusions with the Dirichlet boundary conditions. The main feature of these approximations is their uniformity with respect to the independent variables. The asymptotic formulae are supplied with rigorous remainder estimates. Finally, we offer examples, where results of asymptotic approximations are compared with accurate finite element numerical simulations, and demonstrate the advantages of the asymptotic method.

Keywords: Green's functions, uniform asymptotics, singularly perturbed domains, compound asymptotic expansions.

1 Introduction

The study of Green's functions in domains with perturbed boundaries was initiated by the classical work of Hadamard (see [5]) who analyzed Green's kernels both for the Laplacian and the biharmonic problem in a domain with a regularly perturbed smooth boundary. The asymptotic formula derived in

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[5] is often referred to as the Hadamard variational formula, and it had substantial impact on several areas of the theory of partial differential equations: among its applications are shape sensitivity and optimization analysis [3], free boundary problems [14], Brownian motion on hypersurfaces [6]. Analogues of Hadamard's formula were also obtained for general elliptic boundary value problems [4] as well as for the heat equation. Neither Hadamard's formula nor asymptotic approximations related to the above mentioned applications are uniform with respect to independent variables. The correction, which makes Hadamard's approximation uniform, was recently stated in [9].

The paper [8] includes uniform asymptotic approximations of Green's functions of Dirichlet boundary value problems for the Laplacian in domains with small inclusions. Analysis of other types of boundary conditions and uniform asymptotics of Green's functions for domains of different shapes (singularly perturbed cones, thin cylinders etc.) were published in [9]. In both papers [8], [9], we employ the method of compound asymptotic expansions (see [11]).

The asymptotic analysis of [8, 9] has been extended in [10] to Green's tensors of vector elasticity, in both two and three dimensions, for a solid containing a small inclusion. The asymptotic approximations are uniform, and the paper [10] also includes the rigorous remainder estimates.

In the present paper, the earlier results of [8], [10] are further advanced to problems of elasticity for solids with multiple inclusions. In addition to the analytical formulae, we also include the numerical simulations illustrating the efficiency of the asymptotic approximations.

The structure of the article is as follows.

Section 2 contains the main notations adopted throughout the text and the formulation of Green's function for the operator $-\Delta$ in a planar domain with several inclusions (the case of anti-plane shear). In Section 3, we deal with the asymptotic approximation of the Green's function for anti-plane shear in a planar region with several inclusions. In Section 4, we show how the asymptotic formula for Green's function simplifies under constraints on the independent spatial variables within the singularly perturbed domain. Following the analytical investigation of the approximation of Green's function for $-\Delta$ in a domain with multiple inclusions, in Section 5 we then proceed with the numerical computations to illustrate the efficiency of the asymptotic algorithm. Here we consider the regular part of Green's function for Laplace's operator, in the case of a planar domain with several inclusions. We then extend the theory developed in Section 3 to the case of Green's tensor for the system of elasticity in Section 6. In Section 7, another example is treated where we compare the asymptotic approximation of Section 6 against the benchmark finite element computations. Finally, in Section 8, we construct the approximation of Green's tensor for a three dimensional body with several inclusions, followed by Section 9 with simplified asymptotic formulae (under the constraints of Section 4).

In what follows, G_ε denotes Green's tensor for the planar perturbed bounded domain Ω_ε, ε is a small positive parameter, G is Green's tensor for the unperturbed domain, $g^{(j)}$ is Green's tensor for the unbounded domain corresponding to the j^{th} inclusion and γ is the fundamental solution of the Lamé operator in two dimensions. We also have $\zeta^{(j)}$ as the limit of $g^{(j)}$ at infinity, $\zeta^{(\infty,j)}$ is a constant matrix present in the asymptotics of $\zeta^{(j)}$ at infinity. In addition, we make use of the elastic capacitary potential $P_\varepsilon^{(j)}$ related to the j^{th} inclusion, defined in the perturbed domain. The notation $\mathbf{y}$ denotes the position of the point force, $\mathbf{x}$ is the spatial variable where the measurement of displacement produced by the force at $\mathbf{y}$ is to be taken and $\mathbf{O}^{(j)}$ is the centre of the j^{th} inclusion. By λ and μ we mean elastic moduli.

As one of our main results presented in this article, we prove the following

Theorem *Green's tensor for the Lamé operator in $\Omega_\varepsilon \subset \mathbb{R}^2$ admits the representation*

$$
\begin{aligned}
G_\varepsilon(\mathbf{x},\mathbf{y}) &= G(\mathbf{x},\mathbf{y}) + \sum_{j=1}^{N} g^{(j)}\left(\frac{x-\mathbf{O}^{(j)}}{\varepsilon}, \frac{\mathbf{y}-\mathbf{O}^{(j)}}{\varepsilon}\right) - N\gamma\left(\frac{\mathbf{x}}{\varepsilon}, \frac{\mathbf{y}}{\varepsilon}\right) \\
&+ \sum_{j=1}^{N}\left\{P_\varepsilon^{(j)}(\mathbf{x})A^{(j)}P_\varepsilon^{(j)T}(\mathbf{y}) - \zeta^{(j)}\left(\frac{\mathbf{x}-\mathbf{O}^{(j)}}{\varepsilon}\right) - \zeta^{(j)}\left(\frac{\mathbf{y}-\mathbf{O}^{(j)}}{\varepsilon}\right) + \zeta^{(\infty,j)}\right\} \\
&- \sum_{j=1}^{N}\sum_{\substack{k\neq j\\ 1\le k\le N}} P_\varepsilon^{(j)}(\mathbf{x})G(\mathbf{O}^{(j)},\mathbf{O}^{(k)})P_\varepsilon^{(k)T}(\mathbf{y}) + O(\varepsilon)\ ,
\end{aligned} \tag{1}
$$

uniformly with respect to $(\mathbf{x},\mathbf{y}) \in \Omega_\varepsilon \times \Omega_\varepsilon$*, where*

$$A^{(j)} = (\lambda+3\mu)(4\pi\mu(\lambda+2\mu))^{-1}\log\varepsilon I_2 + H(\mathbf{O}^{(j)},\mathbf{O}^{(j)}) - \zeta^{(\infty,j)}\ , \quad 1 \le j \le N\ .$$

In the above theorem, the last term in (1) denotes a matrix whose components are $O(\varepsilon)$.

The regular part of Green's function in a disk with 5 circular inclusions is shown in Fig 1 for the case of anti-plane shear; here $\varepsilon = 0.2974$ and the asymptotic approximation of the regular part for Green's function is compared with the corresponding numerical solution obtained in FEMLAB. It can be seen that both plots are very similar; the maximum absolute error here is as small as 0.0206.

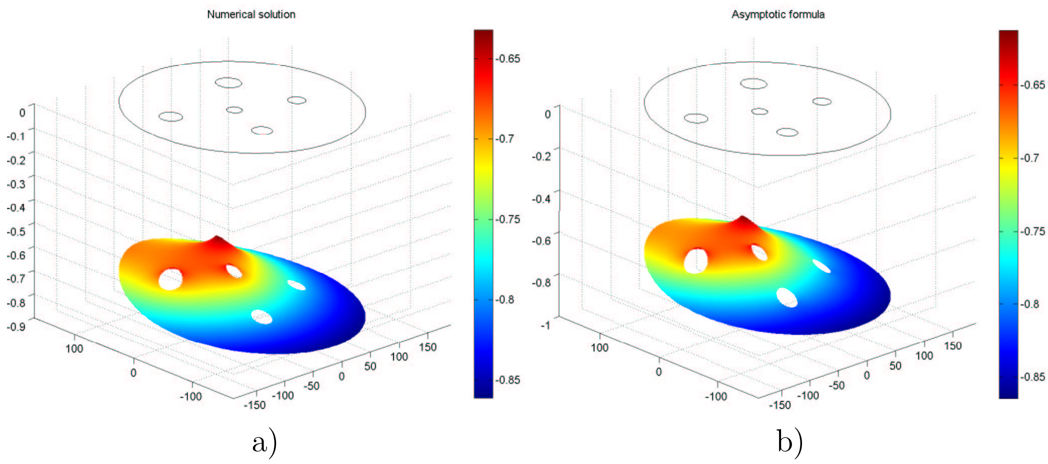


Figure 1: a) Numerical solution produced in FEMLAB, b) Computations produced by the asymptotic approximation for the regular part of Green's function for $\varepsilon = 0.2974$.

2 Main notations and governing equations

We now give several notations adopted in the following text. Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2, 3$ with compact closure $\bar{\Omega}$ and smooth boundary $\partial\Omega$. By $\omega^{(j)}$, $j = 1, \dots, N$, we denote domains in $\mathbb{R}^n$ with smooth boundary $\partial\omega^{(j)}$ and compact closure $\bar{\omega}^{(j)}$; its complement being $C\bar{\omega}^{(j)} = \mathbb{R}^n \backslash \bar{\omega}^{(j)}$. We shall assume that $\omega^{(j)}$, $j = 1, \dots, N$ contains the origin $\mathbf{O}$ as an interior point. We introduce the sets $\omega_\varepsilon^{(j)} = \{\mathbf{x} : \varepsilon^{-1}(\mathbf{x} - \mathbf{O}^{(j)}) \in \omega^{(j)}\}$, where ε is a small positive parameter and $\mathbf{O}^{(j)}$ being the centre of $\omega_\varepsilon^{(j)}$. Also we have the open set $\Omega_\varepsilon = \Omega \backslash \bigcup_j \bar{\omega}_\varepsilon^{(j)}$. It is also assumed that the minimum distance between $\mathbf{O}^{(j)}$ and the points of $\partial\Omega$ is equal to 1. In addition the maximum distance between $\mathbf{O}$ and the points of $\partial\omega^{(j)}$ will be taken as 1.

The main object of our study in Sections 3-5 is Green's function for the Laplacian in $\Omega_\varepsilon \subset \mathbb{R}^2$, will be denoted by G_ε. The function G_ε is a solution of

$$-\Delta_{\mathbf{x}} G_\varepsilon(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) \;, \quad \mathbf{x}, \mathbf{y} \in \Omega_\varepsilon \;, \tag{2}$$

$$G_\varepsilon(\mathbf{x}, \mathbf{y}) = 0 \;, \quad \mathbf{x} \in \partial\Omega_\varepsilon, \mathbf{y} \in \Omega_\varepsilon \;. \tag{3}$$

In the sequel, along with $\mathbf{x}$ and $\mathbf{y}$, we shall use scaled variables $\boldsymbol{\xi}_j = \varepsilon^{-1}(\mathbf{x} - \mathbf{O}^{(j)})$ and $\boldsymbol{\eta}_j = \varepsilon^{-1}(\mathbf{y} - \mathbf{O}^{(j)})$, $j = 1, \dots, N$.

By const we always mean different positive constants, from Section 6 onwards this constant may depend on the elastic moduli λ and μ. The notation $f = O(g)$ for a scalar function f is equivalent to the inequality

$|f| \le \text{const}\, g$. Whenever we write $f = O(g)$ for a matrix (vector) function f, we mean a matrix (vector) f whose components are $O(g)$.

3 Green's function for the case of anti-plane shear for a domain with several inclusions

Let $G(\mathbf{x}, \mathbf{y})$ and $g^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j)$ denote Green's function for the operator $-\Delta$ in the domains Ω and $C\bar{\omega}^{(j)}$, $j = 1, \dots, N$, respectively. The function G is a solution the following problem

$$-\Delta_{\mathbf{x}} G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) \;, \quad \mathbf{x}, \mathbf{y} \in \Omega \;, \tag{4}$$

$$G(\mathbf{x}, \mathbf{y}) = 0 \;, \quad \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega \;, \tag{5}$$

and the functions $g^{(j)}$ solve

$$-\Delta_{\boldsymbol{\xi}_j} g^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) = \delta(\boldsymbol{\xi}_j - \boldsymbol{\eta}_j) \;, \quad \boldsymbol{\xi}_j, \boldsymbol{\eta}_j \in C\bar{\omega}^{(j)} \;, \tag{6}$$

$$g^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) = 0 \;, \quad \boldsymbol{\xi}_j \in \partial C\bar{\omega}^{(j)}, \boldsymbol{\eta}_j \in C\bar{\omega}^{(j)} \;, \tag{7}$$

$$g^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) \text{ is bounded as } |\boldsymbol{\xi}_j| \to \infty \;, \boldsymbol{\eta}_j \in C\bar{\omega}^{(j)} \;. \tag{8}$$

We represent $G(\mathbf{x}, \mathbf{y})$ as

$$G(\mathbf{x}, \mathbf{y}) = -(2\pi)^{-1} \log |\mathbf{x} - \mathbf{y}| - H(\mathbf{x}, \mathbf{y}) \;, \tag{9}$$

and $g^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j)$ for $j = 1, \dots, N$ as

$$g^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) = -(2\pi)^{-1} \log |\boldsymbol{\xi}_j - \boldsymbol{\eta}_j| - h^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) \;, \tag{10}$$

where H and $h^{(j)}$ are the regular parts of G and $g^{(j)}$ respectively, and the first term in the right-hand sides of (9) and (10) is the fundamental solution of the operator $-\Delta$.

We introduce the function $\zeta^{(j)}$ as

$$\zeta^{(j)}(\boldsymbol{\eta}_j) = \lim_{|\boldsymbol{\xi}_j| \to \infty} g^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) \;, \tag{11}$$

and the constant

$$\zeta^{(j)}_\infty = \lim_{|\boldsymbol{\eta}_j| \to \infty} \{\zeta^{(j)}(\boldsymbol{\eta}_j) - (2\pi)^{-1} \log |\boldsymbol{\eta}_j|\} \;, \tag{12}$$

for $j = 1, \dots, N$.

3.1 Auxiliary functions

3.1.1 Estimates for the functions $h^{(j)}$ and $\zeta^{(j)}$ in the unbounded domain

In this subsection we state two results related to the functions $h^{(j)}$ and $\zeta^{(j)}$, $j = 1, \dots, N$, which will be used in the algorithm for the asymptotic expansion of the function G_ε.

The proof of the following two lemmas can be found in [8].

Lemma 1 *For $|\boldsymbol{\xi}_j| > 2$ and $\boldsymbol{\eta}_j \in C\bar{\omega}^{(j)}$ the following estimate holds*

$$h^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) = -(2\pi)^{-1} \log |\boldsymbol{\xi}_j| - \zeta^{(j)}(\boldsymbol{\eta}_j) + O(|\boldsymbol{\xi}_j|^{-1}) \,, \tag{13}$$

for $j = 1, \dots, N$.

Lemma 2 *For $|\boldsymbol{\xi}_j| > 2$, the following representation for $\zeta^{(j)}$ holds*

$$\zeta^{(j)}(\boldsymbol{\xi}_j) = (2\pi)^{-1} \log |\boldsymbol{\xi}_j| + \zeta^{(j)}_\infty + O(|\boldsymbol{\xi}_j|^{-1}) \,, \tag{14}$$

for $j = 1, \dots, N$.

3.1.2 The equilibrium potential

Let $P^{(j)}_\varepsilon(\mathbf{x})$ be the equilibrium potential corresponding to the j^{th} inclusion with centre $\mathbf{O}^{(j)}$. The function $P^{(j)}_\varepsilon(\mathbf{x})$ is defined as a solution of

$$\Delta P^{(j)}_\varepsilon(\mathbf{x}) = 0 \,, \quad \mathbf{x} \in \Omega_\varepsilon \,, \tag{15}$$

$$P^{(j)}_\varepsilon(\mathbf{x}) = 0 \,, \quad \mathbf{x} \in \partial\Omega \,, \tag{16}$$

$$P^{(j)}_\varepsilon(\mathbf{x}) = \delta_{ij} \,, \quad \mathbf{x} \in \partial\omega^{(i)}_\varepsilon, i = 1, \dots, N \,. \tag{17}$$

where δ_{ij} is the Kronecker delta.

We give a uniform approximation of the function $P^{(j)}_\varepsilon$, by considering the vector $P_\varepsilon(\mathbf{x}) = \{P^{(j)}_\varepsilon(\mathbf{x})\}_{j=1}^N$

Theorem 1 *The asymptotic approximation of $P_\varepsilon(\mathbf{x})$ is given by the formula,*

$$P_\varepsilon(\mathbf{x}) = \left(\operatorname*{diag}_{1 \le j \le N} \{\alpha^{(j)}_\varepsilon\} - \mathfrak{M} \right)^{-1} \mathcal{S}(\mathbf{x}) + p_\varepsilon(\mathbf{x}) \tag{18}$$

where $\alpha^{(j)}_\varepsilon = (2\pi)^{-1} \log \varepsilon + H(\mathbf{O}^{(j)}, \mathbf{O}^{(j)}) - \zeta^{(j)}_\infty$, $\mathfrak{M} = \{(1-\delta_{kj}) G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)})\}_{k,j=1}^N$, $\mathcal{S}(\mathbf{x}) = \{-G(\mathbf{x}, \mathbf{O}^{(j)}) + \zeta^{(j)}(\boldsymbol{\xi}_j) - (2\pi)^{-1} \log |\boldsymbol{\xi}_j| - \zeta^{(j)}_\infty\}_{j=1}^N$, and the vector $p_\varepsilon(\mathbf{x})$ is the remainder term such that

$$|p_\varepsilon(\mathbf{x})| \le \text{const}\, \varepsilon (\log \varepsilon)^{-2}, \tag{19}$$

uniformly with respect to $\mathbf{x} \in \Omega_\varepsilon$.

Prior to the proof of Theorem 1 we shall show that the leading order term of the functions $P_\varepsilon^{(j)}$ are solutions of a certain algebraic system.

Lemma 3 *The leading order part $\mathcal{P}_\varepsilon^{(j)}$ of the functions $P_\varepsilon^{(j)}$ are solutions of*

$$\left(\operatorname*{diag}_{1\le j\le N}\{\alpha_\varepsilon^{(j)}\} - \mathfrak{M}\right)\mathcal{P}_\varepsilon(\mathbf{x}) = \mathcal{S}(\mathbf{x})\ , \tag{20}$$

where $\mathcal{P}_\varepsilon = \{\mathcal{P}_\varepsilon^{(j)}\}_{j=1}^N$.

Proof. We represent $P_\varepsilon^{(j)}(\mathbf{x})$ in the form

$$P_\varepsilon^{(j)}(\mathbf{x}) = \frac{-G(\mathbf{x},\mathbf{O}^{(j)}) + \zeta^{(j)}(\boldsymbol{\xi}_j) - (2\pi)^{-1}\log|\boldsymbol{\xi}_j| - \zeta_\infty^{(j)}}{(2\pi)^{-1}\log\varepsilon + H(\mathbf{O}^{(j)},\mathbf{O}^{(j)}) - \zeta_\infty^{(j)}} + R_\varepsilon^{(j)}(\mathbf{x})\ , 1\le j\le N\ , \tag{21}$$

where the function $R_\varepsilon^{(j)}(\mathbf{x})$ satisfies

$$\Delta R_\varepsilon^{(j)}(\mathbf{x}) = 0, \quad \mathbf{x}\in\Omega_\varepsilon\ , \tag{22}$$

$$R_\varepsilon^{(j)}(\mathbf{x}) = -\frac{\zeta^{(j)}(\boldsymbol{\xi}_j) - (2\pi)^{-1}\log|\boldsymbol{\xi}_j| - \zeta_\infty^{(j)}}{(2\pi)^{-1}\log\varepsilon + H(\mathbf{O}^{(j)},\mathbf{O}^{(j)}) - \zeta_\infty^{(j)}}\ , \quad \mathbf{x}\in\partial\Omega\ , \tag{23}$$

$$R_\varepsilon^{(j)}(\mathbf{x}) = 1 - \frac{(2\pi)^{-1}\log\varepsilon + H(\mathbf{x},\mathbf{O}^{(j)}) - \zeta_\infty^{(j)}}{(2\pi)^{-1}\log\varepsilon + H(\mathbf{O}^{(j)},\mathbf{O}^{(j)}) - \zeta_\infty^{(j)}}\ , \quad \mathbf{x}\in\partial\omega_\varepsilon^{(j)}\ , \tag{24}$$

$$\begin{aligned} R_\varepsilon^{(j)}(\mathbf{x}) \ &= \ \frac{G(\mathbf{x},\mathbf{O}^{(j)}) - \zeta^{(j)}(\boldsymbol{\xi}_j) + (2\pi)^{-1}\log|\boldsymbol{\xi}_j| + \zeta_\infty^{(j)}}{(2\pi)^{-1}\log\varepsilon + H(\mathbf{O}^{(j)},\mathbf{O}^{(j)}) - \zeta_\infty^{(j)}}\ , \\ &\quad \mathbf{x}\in\partial\omega_\varepsilon^{(k)}\ , 1\le k\le N\ , k\ne j\ . \end{aligned} \tag{25}$$

The boundary condition (24) is equivalent to

$$R_\varepsilon^{(j)}(\mathbf{x}) = -\frac{H(\mathbf{x},\mathbf{O}^{(j)}) - H(\mathbf{O}^{(j)},\mathbf{O}^{(j)})}{(2\pi)^{-1}\log\varepsilon + H(\mathbf{O}^{(j)},\mathbf{O}^{(j)}) - \zeta_\infty^{(j)}}\ , \quad \mathbf{x}\in\partial\omega_\varepsilon^{(j)}\ , \tag{26}$$

so $R_\varepsilon^{(j)}(\mathbf{x}) = O(\varepsilon(\log\varepsilon)^{-1})$ for $\mathbf{x}\in\partial\omega_\varepsilon^{(j)}$. Using the asymptotic approximation of $\zeta^{(j)}(\boldsymbol{\xi}_j)$ given in Lemma 2, we have from (23) that $R_\varepsilon^{(j)}(\mathbf{x}) = O(\varepsilon(\log\varepsilon)^{-1})$ for $\mathbf{x}\in\partial\Omega$. Then from (25), also using Lemma 2 and the fact $G(\mathbf{x},\mathbf{O}^{(j)})$ is smooth in Ω_ε, we have

$$R_\varepsilon^{(j)}(\mathbf{x}) = \frac{G(\mathbf{O}^{(k)},\mathbf{O}^{(j)})}{(2\pi)^{-1}\log\varepsilon + H(\mathbf{O}^{(j)},\mathbf{O}^{(j)}) - \zeta_\infty^{(j)}} + O(\varepsilon(\log\varepsilon)^{-1})\ , \tag{27}$$

for $\mathbf{x} \in \partial\omega_\varepsilon^{(k)}$, $1 \le k \le N$, $k \ne j$.

Then we may write $R_\varepsilon^{(j)}(\mathbf{x})$, using the equilibrium potential $P_\varepsilon^{(k)}$, $k \ne j$, as

$$R_\varepsilon^{(j)}(\mathbf{x}) = \frac{\sum_{\substack{k\ne j\\ 1\le k\le N}} G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)}) P_\varepsilon^{(k)}(\mathbf{x})}{(2\pi)^{-1}\log\varepsilon + H(\mathbf{O}^{(j)}, \mathbf{O}^{(j)}) - \zeta_\infty^{(j)}} + \mathfrak{p}_\varepsilon^{(j)}(\mathbf{x}) , \tag{28}$$

where $\alpha_\varepsilon^{(j)}$ is as in the formulation of Theorem 1, $\mathfrak{p}_\varepsilon^{(j)}(\mathbf{x})$ is the remainder term.

Now combining (28) with (21), we obtain the following

$$\begin{aligned} P_\varepsilon^{(j)}(\mathbf{x}) \;\; = \;\; & \Big(- G(\mathbf{x}, \mathbf{O}^{(j)}) + \zeta^{(j)}(\boldsymbol{\xi}_j) - (2\pi)^{-1}\log|\boldsymbol{\xi}_j| - \zeta_\infty^{(j)} \\ & + \sum_{\substack{k\ne j\\ 1\le k\le N}}^{N} G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)}) P_\varepsilon^{(k)}(\mathbf{x}) \Big) (\alpha_\varepsilon^{(j)})^{-1} + \mathfrak{p}_\varepsilon^{(j)}(\mathbf{x}) , \end{aligned} \tag{29}$$

where we have $\mathfrak{p}_\varepsilon^{(j)}(\mathbf{x})$ is a function which is harmonic in Ω_ε, is $O(\varepsilon(\log\varepsilon)^{-1})$ for $\mathbf{x} \in \partial\Omega$ and $\mathbf{x} \in \partial\omega_\varepsilon^{(j)}$, $1 \le j \le N$. Therefore by the maximum principle $\mathfrak{p}_\varepsilon^{(j)}(\mathbf{x}) = O(\varepsilon(\log\varepsilon)^{-1})$ for $\mathbf{x} \in \Omega_\varepsilon$.

Then, (29) gives us the following system of algebraic equations in terms of the functions $P_\varepsilon^{(j)}$, whose solution will give us the approximation of the functions $P_\varepsilon^{(j)}$,

$$\Big(\operatorname*{diag}_{1\le j\le N} \{\alpha_\varepsilon^{(j)}\} - \mathfrak{M} \Big) P_\varepsilon(\mathbf{x}) = \mathcal{S}(\mathbf{x}) + \mathfrak{p}_\varepsilon , \tag{30}$$

where $P_\varepsilon(\mathbf{x}) = \{P_\varepsilon^{(j)}(\mathbf{x})\}_{j=1}^N$ and $\mathcal{S}$ and $\mathfrak{M}$ are as in the formulation of Theorem 1, and $\mathfrak{p}_\varepsilon = \{\mathfrak{p}_\varepsilon^{(j)}\}_{j=1}^N$. The leading order part of (30) is equivalent to (20). $\square$

Let

$$\Xi = \Big(\operatorname*{diag}_{1\le j\le N} \{\alpha_\varepsilon^{(j)}\} - \mathfrak{M} \Big)^{-1} , \tag{31}$$

and Ξ_{ij}, $i, j = 1, \dots, N$ denote the components of this matrix. Multiplying both sides of (30) by Ξ, we have

$$P_\varepsilon(\mathbf{x}) = \Xi\mathcal{S}(\mathbf{x}) + p_\varepsilon , \tag{32}$$

where $p_\varepsilon = \Xi\mathfrak{p}_\varepsilon$ is the remainder. We shall now estimate the remainder in (32).

The proof of Theorem 1 is given via estimation of the remainder term p_ε. For the estimate of the norm of the vector $p_\varepsilon(\mathbf{x})$ in (18), we shall need an estimate the entries Ξ_{ij} of the matrix Ξ, which is contained in the following Lemma.

Lemma 4 *For the matrix* $\Xi = [\Xi_{ij}]_{i,j=1}^N$*, we have*

$$\Xi_{ij} = \begin{cases} O((\log\varepsilon)^{-1}) & \textit{for } i = j\ , \\ O((\log\varepsilon)^{-2}) & \textit{for } i \neq j\ . \end{cases}$$

Proof. Since $\mathfrak{M}$ is a symmetric matrix, it follows from (31) that Ξ is also symmetric. We have

$$\Xi = (\det(\Xi^{-1}))^{-1}\,\mathrm{adj}(\Xi^{-1})\ , \tag{33}$$

where $\det(\Xi^{-1})$ is the determinant of the $N \times N$ matrix Ξ^{-1} and $\mathrm{adj}(\Xi^{-1})$ is the adjoint of the matrix Ξ^{-1}. Let the matrix of cofactors for Ξ^{-1} be denoted by C with entries

$$C_{ij} = (-1)^{i+j}T_{ij}, \quad i, j = 1, \ldots, N\ ,$$

where T_{ij} are the corresponding minors of Ξ^{-1}.

First, we consider T_{ij} when $i = j$. In this case we shall need to compute the determinant of an $(N-1)\times(N-1)$ matrix, with $N-1$ terms each of $O(\log\varepsilon)$ along the diagonal, and with off-diagonal components of $O(1)$. Thus T_{ij} for $i = j$ is then is $O((\log\varepsilon)^{N-1})$.

Next consider T_{ij}, when $i \neq j$, so that we compute the determinant of an $(N-1)\times(N-1)$ matrix, containing $N-2$ components of $O(\log\varepsilon)$ and all other components of $O(1)$. Then T_{ij}, for $i \neq j$ is $O((\log\varepsilon)^{N-2})$. Therefore

$$C_{ij} = \begin{cases} O((\log\varepsilon)^{N-1}) & \text{for } i = j\ , \\ O((\log\varepsilon)^{N-2}) & \text{for } i \neq j\ . \end{cases}$$

Since $\det(\Xi^{-1})$ is $O((\log\varepsilon)^N)$ we complete the proof of the Lemma. □

Now, we finalize the proof of Theorem 1

Proof of Theorem 2. The asymptotic approximation of the vector P_ε admits the representation given in (32) as a consequence of Lemma 3, with the remainder term given by $p_\varepsilon = \Xi\mathfrak{p}_\varepsilon$. In the proof of Lemma 3, it was shown that the vector $\mathfrak{p}_\varepsilon$ has entries which are $O(\varepsilon(\log\varepsilon)^{-1})$, thus by the preceding Lemma, we have the remainder term p_ε has the vector norm $|p_\varepsilon| = O(\varepsilon(\log\varepsilon)^{-2})$. The proof of Theorem 1 is complete. □

3.2 A uniform asymptotic approximation of Green's function for $-\Delta$ in a 2-dimensional domain with several small inclusions

Now we may approach the approximation of Green's matrix G_ε for the Laplacian in a planar domain with several inclusions.

Theorem 2 *Green's function for the operator* $-\Delta$ *in* $\Omega_\varepsilon \subset \mathbb{R}^2$ *admits the representation*

$$\begin{aligned} G_\varepsilon(\mathbf{x},\mathbf{y}) &= G(\mathbf{x},\mathbf{y}) + \sum_{j=1}^{N} g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) + N(2\pi)^{-1}\log(\varepsilon^{-1}|\mathbf{x}-\mathbf{y}|) \\ &\quad + \sum_{j=1}^{N}\{\alpha_\varepsilon^{(j)} P_\varepsilon^{(j)}(\mathbf{y}) P_\varepsilon^{(j)}(\mathbf{x}) - \zeta^{(j)}(\boldsymbol{\xi}_j) - \zeta^{(j)}(\boldsymbol{\eta}_j) + \zeta_\infty^{(j)}\} \\ &\quad - \sum_{j=1}^{N} \sum_{\substack{k\neq j \\ 1\leq k\leq N}} G(\mathbf{O}^{(k)},\mathbf{O}^{(j)}) P_\varepsilon^{(k)}(\mathbf{y}) P_\varepsilon^{(j)}(\mathbf{x}) + O(\varepsilon)\,, \end{aligned} \tag{34}$$

uniformly with respect to $(\mathbf{x},\mathbf{y}) \in \Omega_\varepsilon \times \Omega_\varepsilon$.

Proof. For this we propose that G_ε may be given as

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = -(2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}| - H_\varepsilon(\mathbf{x},\mathbf{y}) - \sum_{j=1}^{N} h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})\,, \tag{35}$$

where it suffices to seek the approximation of the functions $H_\varepsilon(\mathbf{x},\mathbf{y})$ and $h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})$, which are solutions of the problems

$$\Delta_{\mathbf{x}} H_\varepsilon(\mathbf{x},\mathbf{y}) = 0\,, \quad \mathbf{x},\mathbf{y} \in \Omega_\varepsilon\,, \tag{36}$$

$$H_\varepsilon(\mathbf{x},\mathbf{y}) = -(2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|\,, \quad \mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega_\varepsilon\,, \tag{37}$$

$$H_\varepsilon(\mathbf{x},\mathbf{y}) = 0\,, \quad \mathbf{x}\in\partial\omega_\varepsilon^{(j)}, \mathbf{y}\in\Omega_\varepsilon, 1\leq j\leq N\,, \tag{38}$$

and

$$\Delta_{\mathbf{x}} h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = 0\,, \quad \mathbf{x},\mathbf{y}\in\Omega_\varepsilon\,, \tag{39}$$

$$h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = 0\,, \quad \mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega_\varepsilon\,, \tag{40}$$

$$h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = -(2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|\,, \quad \mathbf{x}\in\partial\omega_\varepsilon^{(j)}, \mathbf{y}\in\Omega_\varepsilon\,, \tag{41}$$

$$h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = 0\,, \quad \mathbf{x}\in\partial\omega_\varepsilon^{(k)}, \mathbf{y}\in\Omega_\varepsilon\,, 1\leq k\leq N\,, k\neq j\,. \tag{42}$$

The approximation of $H_\varepsilon(\mathbf{x},\mathbf{y})$. Let $H_\varepsilon(\mathbf{x},\mathbf{y})$ be given by

$$H_\varepsilon(\mathbf{x},\mathbf{y}) = -H(\mathbf{O}^{(j)},\mathbf{y})P_\varepsilon^{(j)}(\mathbf{x}) + H(\mathbf{x},\mathbf{y}) + V(\mathbf{x},\mathbf{y}) \;, \tag{43}$$

where $V(\mathbf{x},\mathbf{y})$ satisfies

$$\Delta_{\mathbf{x}} V(\mathbf{x},\mathbf{y}) = 0 \;, \quad \mathbf{x},\mathbf{y} \in \Omega_\varepsilon \;, \tag{44}$$

$$V(\mathbf{x},\mathbf{y}) = 0 \;, \quad \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega_\varepsilon \;, \tag{45}$$

$$V(\mathbf{x},\mathbf{y}) = H(\mathbf{O}^{(j)},\mathbf{y}) - H(\mathbf{x},\mathbf{y}) \;, \quad \mathbf{x} \in \partial\omega_\varepsilon^{(j)}, \mathbf{y} \in \Omega_\varepsilon \;, \tag{46}$$

$$V(\mathbf{x},\mathbf{y}) = -H(\mathbf{x},\mathbf{y}) \;, \quad \mathbf{x} \in \partial\omega_\varepsilon^{(k)}, \mathbf{y} \in \Omega_\varepsilon \;, k \neq j \;, 1 \leq k \leq N \;. \tag{47}$$

Since $\omega_\varepsilon^{(j)}$, $1 \leq j \leq N$, are small inclusions and H is a smooth function in Ω, we may expand H about the centres of the inclusions. Namely, for the boundary condition (46) we have

$$V(\mathbf{x},\mathbf{y}) = H(\mathbf{O}^{(j)},\mathbf{y}) - H(\mathbf{x},\mathbf{y}) = O(\varepsilon) \;, \quad \mathbf{x} \in \partial\omega_\varepsilon^{(j)}, \mathbf{y} \in \Omega_\varepsilon \;, \tag{48}$$

and from (47)

$$\begin{aligned} V(\mathbf{x},\mathbf{y}) &= -H(\mathbf{x},\mathbf{y}) = -H(\mathbf{O}^{(k)},\mathbf{y}) + O(\varepsilon) \;, \\ &\qquad \mathbf{x} \in \partial\omega_\varepsilon^{(k)}, \mathbf{y} \in \Omega_\varepsilon \;, k \neq j \;, 1 \leq k \leq N \;. \end{aligned} \tag{49}$$

We therefore write the function $V(\mathbf{x},\mathbf{y})$ as

$$V(\mathbf{x},\mathbf{y}) = -\sum_{\substack{k\neq j\\ 1\leq k\leq N}} H(\mathbf{O}^{(k)},\mathbf{y})P_\varepsilon^{(k)}(\mathbf{x}) + \mathfrak{H}_\varepsilon(\mathbf{x},\mathbf{y}) \;, \tag{50}$$

where $\mathfrak{H}_\varepsilon$ is the remainder term. Substituting (50) into (43) we have

$$H_\varepsilon(\mathbf{x},\mathbf{y}) = -\sum_{j=1}^{N} H(\mathbf{O}^{(j)},\mathbf{y})P_\varepsilon^{(j)}(\mathbf{x}) + H(\mathbf{x},\mathbf{y}) + \mathfrak{H}_\varepsilon(\mathbf{x},\mathbf{y}) \;, \tag{51}$$

where $\mathfrak{H}_\varepsilon(\mathbf{x},\mathbf{y})$ satisfies

$$\Delta_{\mathbf{x}} \mathfrak{H}_\varepsilon(\mathbf{x},\mathbf{y}) = 0 \;, \quad \mathbf{x},\mathbf{y} \in \Omega_\varepsilon \;, \tag{52}$$

$$\mathfrak{H}_\varepsilon(\mathbf{x},\mathbf{y}) = 0 \;, \quad \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega_\varepsilon \;, \tag{53}$$

$$\begin{aligned} \mathfrak{H}_\varepsilon(\mathbf{x},\mathbf{y}) &= H(\mathbf{O}^{(j)},\mathbf{y}) - H(\mathbf{x},\mathbf{y}) \\ &= O(\varepsilon) \;, \quad \mathbf{x} \in \partial\omega_\varepsilon^{(j)}, \mathbf{y} \in \Omega_\varepsilon, 1 \leq j \leq N \;, \end{aligned} \tag{54}$$

and therefore by the maximum principle $\mathfrak{H}_\varepsilon(\mathbf{x},\mathbf{y}) = O(\varepsilon)$, uniformly with respect to $\mathbf{x},\mathbf{y} \in \Omega_\varepsilon$.

The approximation of $h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})$. We begin by writing the boundary condition (41) on $\partial\omega_\varepsilon^{(j)}$ as

$$h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = -(2\pi)^{-1}\log\varepsilon - (2\pi)^{-1}\log(\varepsilon^{-1}|\mathbf{x}-\mathbf{y}|) \ , \quad \mathbf{x} \in \partial\omega_\varepsilon^{(j)} \ , \mathbf{y} \in \Omega_\varepsilon \ . \tag{55}$$

We seek $h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})$ in the form

$$h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = -(2\pi)^{-1}\log\varepsilon + h^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) + \chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) \ , \tag{56}$$

where the remainder $\chi_\varepsilon^{(j)}$ satisfies

$$\Delta_{\mathbf{x}}\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = 0 \ , \quad \mathbf{x},\mathbf{y} \in \Omega_\varepsilon \ , \tag{57}$$

$$\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = (2\pi)^{-1}\log\varepsilon - h^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) \ , \quad \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega_\varepsilon \ , \tag{58}$$

$$\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = 0 \ , \quad \mathbf{x} \in \partial\omega_\varepsilon^{(j)}, \mathbf{y} \in \Omega_\varepsilon \ , \tag{59}$$

$$\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = (2\pi)^{-1}\log\varepsilon - h^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) \ , \quad \mathbf{x} \in \partial\omega_\varepsilon^{(k)}, \mathbf{y} \in \Omega_\varepsilon \ , 1 \le k \le N \ , k \ne j \ . \tag{60}$$

From Lemma 1, we may write boundary conditions (58) and (60) as

$$\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = (2\pi)^{-1}\log|\mathbf{x}-\mathbf{O}^{(j)}| + \zeta^{(j)}(\boldsymbol{\eta}_j) + O(\varepsilon) \ , \quad \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega_\varepsilon \ , \tag{61}$$

$$\begin{aligned} \chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) \ &= \ (2\pi)^{-1}\log|\mathbf{x}-\mathbf{O}^{(j)}| + \zeta^{(j)}(\boldsymbol{\eta}_j) + O(\varepsilon) \ , \\ &\quad \text{for } \mathbf{x} \in \partial\omega_\varepsilon^{(k)}, \mathbf{y} \in \Omega_\varepsilon \ , 1 \le k \le N \ , k \ne j \ . \end{aligned} \tag{62}$$

Then we represent $\chi_\varepsilon^{(j)}$ as

$$\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = -H(\mathbf{x},\mathbf{O}^{(j)}) + (1 - P_\varepsilon^{(j)}(\mathbf{x}))\zeta^{(j)}(\boldsymbol{\eta}_j) + \mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) \ , \tag{63}$$

where $\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})$ satisfies

$$\Delta_{\mathbf{x}}\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = 0 \ , \quad \mathbf{x},\mathbf{y} \in \Omega_\varepsilon \ , \tag{64}$$

$$\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = O(\varepsilon) \ , \quad \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega_\varepsilon \ , \tag{65}$$

$$\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = H(\mathbf{x},\mathbf{O}^{(j)}) \ , \quad \mathbf{x} \in \partial\omega_\varepsilon^{(j)}, \mathbf{y} \in \Omega_\varepsilon \ , \tag{66}$$

$$\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = -G(\mathbf{x},\mathbf{O}^{(j)}) + O(\varepsilon) \ , \quad \mathbf{x} \in \partial\omega_\varepsilon^{(k)}, \mathbf{y} \in \Omega_\varepsilon \ , 1 \le k \le N \ , k \ne j \ . \tag{67}$$

From the fact that $G(\mathbf{x}, \mathbf{O}^{(j)})$ and its regular part are smooth functions in Ω_ε, we expand these functions about the centres of the small inclusions in such a way that boundary conditions (66) and (67) become

$$\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x}, \mathbf{y}) = H(\mathbf{O}^{(j)}, \mathbf{O}^{(j)}) + O(\varepsilon) \;, \quad \mathbf{x} \in \partial\omega_\varepsilon^{(j)}, \mathbf{y} \in \Omega_\varepsilon \;, \tag{68}$$

$$\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x}, \mathbf{y}) = -G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)}) + O(\varepsilon) \;, \quad \mathbf{x} \in \partial\omega_\varepsilon^{(k)}, \mathbf{y} \in \Omega_\varepsilon \;, 1 \le k \le N \;, k \ne j \;. \tag{69}$$

Then the $\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x}, \mathbf{y})$ is given by

$$\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x}, \mathbf{y}) = H(\mathbf{O}^{(j)}, \mathbf{O}^{(j)}) P_\varepsilon^{(j)}(\mathbf{x}) - \sum_{\substack{k \ne j \\ 1 \le k \le N}} G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)}) P_\varepsilon^{(k)}(\mathbf{x}) + O(\varepsilon) \;. \tag{70}$$

Placing (63) and (70) into (56), we obtain the following approximation of $h_\varepsilon(\mathbf{x}, \mathbf{y})$

$$\begin{aligned} h_\varepsilon^{(j)}(\mathbf{x}, \mathbf{y}) &= -(2\pi)^{-1} \log \varepsilon + h^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) - H(\mathbf{x}, \mathbf{O}^{(j)}) \\ &\quad + (1 - P_\varepsilon^{(j)}(\mathbf{x})) \zeta^{(j)}(\boldsymbol{\eta}_j) + H(\mathbf{O}^{(j)}, \mathbf{O}^{(j)}) P_\varepsilon^{(j)}(\mathbf{x}) \\ &\quad - \sum_{\substack{k \ne j \\ 1 \le k \le N}} G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)}) P_\varepsilon^{(k)}(\mathbf{x}) + O(\varepsilon) \;, \end{aligned} \tag{71}$$

which is uniform with respect to $\mathbf{x}, \mathbf{y} \in \Omega_\varepsilon$.

Combined formula. Now substituting (51), (71) into (35) we obtain

$$\begin{aligned} G_\varepsilon(\mathbf{x}, \mathbf{y}) = &= G(\mathbf{x}, \mathbf{y}) + \sum_{j=1}^{N} g^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) + N(2\pi)^{-1} \log(|\mathbf{x} - \mathbf{y}|) \\ &\quad + \sum_{j=1}^{N} (1 - P_\varepsilon^{(j)}(\mathbf{x}))(H(\mathbf{O}^{(j)}, \mathbf{O}^{(j)}) - \zeta^{(j)}(\boldsymbol{\eta}_j) - H(\mathbf{O}^{(j)}, \mathbf{y})) \\ &\quad + \sum_{j=1}^{N} (H(\mathbf{x}, \mathbf{O}^{(j)}) + H(\mathbf{O}^{(j)}, \mathbf{y}) - H(\mathbf{O}^{(j)}, \mathbf{O}^{(j)})) \\ &\quad + \sum_{j=1}^{N} \sum_{\substack{k \ne j \\ 1 \le k \le N}} G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)}) P_\varepsilon^{(k)}(\mathbf{x}) + O(\varepsilon) \;. \end{aligned} \tag{72}$$

Using the following relation obtained from the approximation of $P_\varepsilon^{(j)}(\mathbf{x})$ (see (29)),

$$\begin{aligned} &(H(\mathbf{O}^{(j)}, \mathbf{O}^{(j)}) - \zeta^{(j)}(\boldsymbol{\eta}_j) - H(\mathbf{O}^{(j)}, \mathbf{y}))(\alpha_\varepsilon^{(j)})^{-1} \\ = \;& 1 - P^{(j)}(\mathbf{y}) + (\alpha_\varepsilon^{(j)})^{-1} \sum_{\substack{k \ne j \\ 1 \le k \le N}} G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)}) P_\varepsilon^{(k)}(\mathbf{y}) + O(\varepsilon (\log \varepsilon)^{-1}) \end{aligned} \tag{73}$$

and substituting into (72), we have

$$\begin{aligned}G_\varepsilon(\mathbf{x},\mathbf{y}) &= G(\mathbf{x},\mathbf{y}) + \sum_{j=1}^N g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) + N(2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}| \\ &+ \sum_{j=1}^N \alpha_\varepsilon^{(j)}(1-P_\varepsilon^{(j)}(\mathbf{x}))(1-P_\varepsilon^{(j)}(\mathbf{y})) \\ &+ \sum_{j=1}^N (H(\mathbf{x},\mathbf{O}^{(j)}) + H(\mathbf{O}^{(j)},\mathbf{y}) - H(\mathbf{O}^{(j)},\mathbf{O}^{(j)})) \\ &+ \sum_{j=1}^N \sum_{\substack{k\neq j\\1\le k\le N}} G(\mathbf{O}^{(k)},\mathbf{O}^{(j)})\{P_\varepsilon^{(k)}(\mathbf{y}) + P_\varepsilon^{(k)}(\mathbf{x}) \\ &- P_\varepsilon^{(k)}(\mathbf{y})P_\varepsilon^{(j)}(\mathbf{x})\} + O(\varepsilon)\ . \qquad (74)\end{aligned}$$

Then, expanding the fourth term on the right-hand side of (74) and using (73), we have

$$\begin{aligned}&\sum_{j=1}^N \alpha_\varepsilon^{(j)}(1-P_\varepsilon^{(j)}(\mathbf{x}))(1-P_\varepsilon^{(j)}(\mathbf{y})) \\ =& -\sum_{j=1}^N (H(\mathbf{x},\mathbf{O}^{(j)}) + H(\mathbf{O}^{(j)},\mathbf{y}) - H(\mathbf{O}^{(j)},\mathbf{O}^{(j)})) \\ &- \sum_{j=1}^N (\zeta^{(j)}(\boldsymbol{\xi}_j) + \zeta^{(j)}(\boldsymbol{\eta}_j) - \zeta_\infty^{(j)}) - N(2\pi)^{-1}\log\varepsilon \\ &- \sum_{j=1}^N \sum_{\substack{k\neq j\\1\le k\le N}} G(\mathbf{O}^{(k)},\mathbf{O}^{(j)})\{P_\varepsilon^{(k)}(\mathbf{y}) + P_\varepsilon^{(k)}(\mathbf{x})\} \\ &+ \sum_{j=1}^N \alpha_\varepsilon^{(j)} P_\varepsilon^{(j)}(\mathbf{y})P_\varepsilon^{(j)}(\mathbf{x}) + O(\varepsilon)\ . \qquad (75)\end{aligned}$$

Substitution of (75) in (74) leads to the formula (34). The proof is complete. □

4 Simplified asymptotic formulae subject to constraints on the independent variables

Here we show how the asymptotic formula for G_ε (see (34)), may be simplified under suitable assumptions on the points $\mathbf{x}, \mathbf{y}$. We consider two cases, the first being the situation when the points $\mathbf{x}, \mathbf{y}$ are sufficiently far away from each of the inclusions, the second is when the points are within a small neighborhood of a particular inclusion.

Corollary 1 *a) Let* $\mathbf{x}, \mathbf{y} \in \Omega_\varepsilon \subset \mathbb{R}^2$ *such that*

$$\min\{|\mathbf{x} - \mathbf{O}^{(j)}|, |\mathbf{y} - \mathbf{O}^{(j)}|\} > 2\,\varepsilon \textit{ for all } j = 1, \dots, N. \tag{76}$$

Then

$$\begin{aligned} G_\varepsilon(\mathbf{x}, \mathbf{y}) \;&=\; G(\mathbf{x}, \mathbf{y}) + \sum_{i=1}^{N} \Xi_{ii} G(\mathbf{y}, \mathbf{O}^{(i)}) G(\mathbf{x}, \mathbf{O}^{(i)}) \\ &\quad + O\left(\sum_{i=1}^{N} \varepsilon (\min\{|\mathbf{x} - \mathbf{O}^{(i)}|, |\mathbf{y} - \mathbf{O}^{(i)}|\})^{-1} \right) , \end{aligned} \tag{77}$$

where $\Xi = [\Xi_{ij}]_{i,j=1}^{2}$*, is given by* (31).

b) If $\max\{|\mathbf{x} - \mathbf{O}^{(m)}|, |\mathbf{y} - \mathbf{O}^{(m)}|\} < 1/2$*, then*

$$\begin{aligned} G_\varepsilon(\mathbf{x}, \mathbf{y}) \;&=\; g^{(m)}(\boldsymbol{\xi}_m, \boldsymbol{\eta}_m) + (\alpha_\varepsilon^{(m)})^{-1} \zeta^{(m)}(\boldsymbol{\eta}_m) \zeta^{(m)}(\boldsymbol{\xi}_m) \\ &\quad + O(\max\{|\mathbf{x} - \mathbf{O}^{(m)}|, |\mathbf{y} - \mathbf{O}^{(m)}|\}) , \end{aligned} \tag{78}$$

where $\alpha_\varepsilon^{(j)} = (2\pi)^{-1} \log \varepsilon + H(\mathbf{O}^{(j)}, \mathbf{O}^{(j)}) - \zeta_\infty^{(j)}$.

Both (77) *and* (78) *are uniform with respect to* $(\mathbf{x}, \mathbf{y}) \in \Omega_\varepsilon \times \Omega_\varepsilon$.

Proof. a) From (34), G_ε may be written as

$$\begin{aligned} G_\varepsilon(\mathbf{x}, \mathbf{y}) \;&=\; G(\mathbf{x}, \mathbf{y}) - \sum_{j=1}^{N} h^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) \\ &\quad + \sum_{j=1}^{N} \left\{ \alpha_\varepsilon^{(j)} P_\varepsilon^{(j)}(\mathbf{y}) P_\varepsilon^{(j)}(\mathbf{x}) - \zeta^{(j)}(\boldsymbol{\xi}_j) - \zeta^{(j)}(\boldsymbol{\eta}_j) + \zeta_\infty^{(j)} \right\} \\ &\quad - \sum_{j=1}^{N} \sum_{\substack{k \neq j \\ 1 \leq k \leq N}} G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)}) P_\varepsilon^{(k)}(\mathbf{y}) P_\varepsilon^{(j)}(\mathbf{x}) + O(\varepsilon) . \end{aligned} \tag{79}$$

Owing to Lemma 2, we have the estimate for the function $\zeta^{(j)}$

$$\zeta^{(j)}(\boldsymbol{\xi}_j) = (2\pi)^{-1} \log |\boldsymbol{\xi}_j| + \zeta_\infty^{(j)} + O(|\boldsymbol{\xi}_j|^{-1}) \;, \tag{80}$$

and, as a result of condition (76), along with the estimate for $h^{(j)}$ given in Lemma 1 we obtain

$$\begin{aligned} h^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) &= -(2\pi)^{-1} \log |\boldsymbol{\xi}_j| - \zeta^{(j)}(\boldsymbol{\eta}_j) + O(|\boldsymbol{\xi}_j|^{-1}) \\ &= -(2\pi)^{-1} \log |\boldsymbol{\xi}_j| - (2\pi)^{-1} \log |\boldsymbol{\eta}_j| - \zeta_\infty^{(j)} \\ &\quad + O(\varepsilon (\min\{|\mathbf{x} - \mathbf{O}^{(j)}|, |\mathbf{y} - \mathbf{O}^{(j)}|\})^{-1}) \;. \end{aligned} \tag{81}$$

Using the latter estimates in (79), yields

$$\begin{aligned} G_\varepsilon(\mathbf{x}, \mathbf{y}) &= G(\mathbf{x}, \mathbf{y}) + \sum_{j=1}^{N} \alpha_\varepsilon^{(j)} P_\varepsilon^{(j)}(\mathbf{y}) P_\varepsilon^{(j)}(\mathbf{x}) \\ &\quad - \sum_{j=1}^{N} \sum_{\substack{k \neq j \\ 1 \leq k \leq N}} G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)}) P_\varepsilon^{(k)}(\mathbf{y}) P_\varepsilon^{(j)}(\mathbf{x}) \\ &\quad + O\Big(\sum_{j=1}^{N} \varepsilon (\min\{|\mathbf{x} - \mathbf{O}^{(j)}|, |\mathbf{y} - \mathbf{O}^{(j)}|\})^{-1} \Big) \;. \end{aligned} \tag{82}$$

The two summands in (82) may be written as

$$\begin{aligned} & \sum_{j=1}^{N} \alpha_\varepsilon^{(j)} P_\varepsilon^{(j)}(\mathbf{y}) P_\varepsilon^{(j)}(\mathbf{x}) - \sum_{j=1}^{N} \sum_{\substack{k \neq j \\ 1 \leq k \leq N}} G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)}) P_\varepsilon^{(k)}(\mathbf{y}) P_\varepsilon^{(j)}(\mathbf{x}) \\ =\;& P_\varepsilon^T(\mathbf{x}) \operatorname*{diag}_{1 \leq j \leq N} \{\alpha_\varepsilon^{(j)}\} P_\varepsilon(\mathbf{y}) - P_\varepsilon^T(\mathbf{x}) \mathfrak{M} P_\varepsilon(\mathbf{y}) \\ =\;& P_\varepsilon^T(\mathbf{x}) \Xi^{-1} P_\varepsilon(\mathbf{y}) \;, \end{aligned} \tag{83}$$

where $P_\varepsilon = \{P_\varepsilon^{(j)}\}_{j=1}^N$, $\mathfrak{M} = \{(1 - \delta_{jk}) G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)})\}_{k,j=1}^N$, and Ξ is given by (31).

From Theorem 1,

$$P_\varepsilon(\mathbf{x}) = \Xi \mathcal{S}(\mathbf{x}) + O(\varepsilon (\log \varepsilon)^{-2}) \;, \tag{84}$$

where $\mathcal{S}(\mathbf{x}) = \{-G(\mathbf{x}, \mathbf{O}^{(j)}) + \zeta^{(j)}(\boldsymbol{\xi}_j) - (2\pi)^{-1} \log |\boldsymbol{\xi}_j| - \zeta_\infty^{(j)}\}_{j=1}^N$, which by Lemma 2, $\mathcal{S}(\mathbf{x}) = \{-G(\mathbf{x}, \mathbf{O}^{(j)}) + O(|\boldsymbol{\xi}_j|^{-1})\}_{j=1}^N$. Then, combining this with

(84) in (83), we may write (83) as

$$\sum_{j=1}^{N} \alpha_\varepsilon^{(j)} P_\varepsilon^{(j)}(\mathbf{x}) P_\varepsilon^{(j)}(\mathbf{y}) - \sum_{j=1}^{N} \sum_{\substack{k\neq j\\ 1\leq k\leq N}} G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)}) P_\varepsilon^{(j)}(\mathbf{x}) P_\varepsilon^{(k)}(\mathbf{y})$$

$$= \sum_{i,m=1}^{N} \Xi_{im} G(\mathbf{y}, \mathbf{O}^{(m)}) G(\mathbf{x}, \mathbf{O}^{(i)})$$

$$+ O\left(\sum_{i=1}^{N} \varepsilon (\min\{|\mathbf{x}-\mathbf{O}^{(i)}|, |\mathbf{y}-\mathbf{O}^{(i)}|\})^{-1} \right), \tag{85}$$

where Ξ_{im}, $i, m = 1, \dots, N$ are the entries of Ξ. Next, substituting (85) into (82) and since by Lemma 4, the off-diagonal components of Ξ are $O((\log\varepsilon)^{-2})$ we arrive at (77).

$b)$ Using the following expression

$$\sum_{j=1}^{N} \alpha_\varepsilon^{(j)} (1 - P_\varepsilon^{(j)}(\mathbf{x}))(1 - P_\varepsilon^{(j)}(\mathbf{y}))$$

$$= \sum_{j=1}^{N} \alpha_\varepsilon^{(j)} \Big\{ 1 - P_\varepsilon^{(j)}(\mathbf{x}) + (\alpha_\varepsilon^{(j)})^{-1} \sum_{\substack{k\neq j\\ 1\leq k\leq N}} G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)}) P_\varepsilon^{(k)}(\mathbf{x}) \Big\}$$

$$\times \Big\{ 1 - P_\varepsilon^{(j)}(\mathbf{y}) + (\alpha_\varepsilon^{(j)})^{-1} \sum_{\substack{l\neq j\\ 1\leq l\leq N}} G(\mathbf{O}^{(l)}, \mathbf{O}^{(j)}) P_\varepsilon^{(k)}(\mathbf{y}) \Big\}$$

$$+ \sum_{j=1}^{N} \sum_{\substack{k\neq j\\ 1\leq k\leq N}} G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)}) \Big\{ P_\varepsilon^{(j)}(\mathbf{y}) P_\varepsilon^{(k)}(\mathbf{x}) + P_\varepsilon^{(k)}(\mathbf{y}) P_\varepsilon^{(j)}(\mathbf{x}) \tag{86}$$

$$- P_\varepsilon^{(k)}(\mathbf{x}) - P_\varepsilon^{(k)}(\mathbf{y}) - (\alpha_\varepsilon^{(j)})^{-1} \sum_{\substack{l\neq j\\ 1\leq l\leq N}} G(\mathbf{O}^{(l)}, \mathbf{O}^{(j)}) P_\varepsilon^{(l)}(\mathbf{y}) P_\varepsilon^{(k)}(\mathbf{x}) \Big\},$$

along with identity (73) and the definition of G and $g^{(j)}$, $j \neq m$, in (74) we have

$$
\begin{aligned}
G_\varepsilon(\mathbf{x},\mathbf{y}) \;=\;& g^{(m)}(\boldsymbol{\xi}_m,\boldsymbol{\eta}_m) - H(\mathbf{x},\mathbf{y}) - \sum_{\substack{j\neq m\\ 1\le j\le N}} h^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) + (N-1)(2\pi)^{-1}\log\varepsilon \\
&+\sum_{j=1}^{N}(\alpha_\varepsilon^{(j)})^{-1}(H(\mathbf{O}^{(j)},\mathbf{O}^{(j)}) - \zeta^{(j)}(\boldsymbol{\eta}_j) - H(\mathbf{y},\mathbf{O}^{(j)})) \\
&\times(H(\mathbf{O}^{(j)},\mathbf{O}^{(j)}) - \zeta^{(j)}(\boldsymbol{\xi}_j) - H(\mathbf{x},\mathbf{O}^{(j)})) \\
&+\sum_{j=1}^{N}(H(\mathbf{x},\mathbf{O}^{(j)}) + H(\mathbf{y},\mathbf{O}^{(j)}) - H(\mathbf{O}^{(j)},\mathbf{O}^{(j)})) \\
&+\sum_{j=1}^{N}\sum_{\substack{k\neq j\\ 1\le k\le N}} G(\mathbf{O}^{(k)},\mathbf{O}^{(j)})\Big\{P_\varepsilon^{(j)}(\mathbf{y})P_\varepsilon^{(k)}(\mathbf{x}) \\
&-\sum_{\substack{l\neq j\\ 1\le l\le N}}(\alpha_\varepsilon^{(j)})^{-1}G(\mathbf{O}^{(l)},\mathbf{O}^{(j)})P_\varepsilon^{(l)}(\mathbf{y})P_\varepsilon^{(k)}(\mathbf{x})\Big\} + O(\varepsilon)\,. \qquad (87)
\end{aligned}
$$

Since $\max\{|\mathbf{x}-\mathbf{O}^{(m)}|, |\mathbf{y}-\mathbf{O}^{(m)}|\} < 1/2$, we may expand $H(\mathbf{x},\mathbf{y})$ about $(\mathbf{O}^{(m)},\mathbf{O}^{(m)})$, this together with estimates (80), (81) for $j \neq m$ leads to

$$
\begin{aligned}
G_\varepsilon(\mathbf{x},\mathbf{y}) \;=\;& g^{(m)}(\boldsymbol{\xi}_m,\boldsymbol{\eta}_m) - \sum_{\substack{j\neq m\\ 1\le j\le N}}\{-(2\pi)^{-1}\log(\varepsilon^{-1}|\mathbf{x}-\mathbf{O}^{(j)}||\mathbf{y}-\mathbf{O}^{(j)}|) - \zeta^{(\infty,j)}\} \\
&+(\alpha_\varepsilon^{(m)})^{-1}(-\zeta^{(m)}(\boldsymbol{\eta}_m) + O(|\mathbf{y}-\mathbf{O}^{(m)}|))(-\zeta^{(m)}(\boldsymbol{\xi}_m) + O(|\mathbf{x}-\mathbf{O}^{(m)}|)) \\
&+\sum_{\substack{j\neq m\\ 1\le j\le N}}(\alpha_\varepsilon^{(j)})^{-1}(\alpha_\varepsilon^{(j)} + G(\mathbf{y},\mathbf{O}^{(j)}))(\alpha_\varepsilon^{(j)} + G(\mathbf{x},\mathbf{O}^{(j)})) \\
&+\sum_{\substack{j\neq m\\ 1\le j\le N}}(H(\mathbf{x},\mathbf{O}^{(j)}) + H(\mathbf{y},\mathbf{O}^{(j)}) - H(\mathbf{O}^{(j)},\mathbf{O}^{(j)})) \\
&+\sum_{j=1}^{N}\sum_{\substack{k\neq j\\ 1\le k\le N}} G(\mathbf{O}^{(k)},\mathbf{O}^{(j)})\Big\{P_\varepsilon^{(j)}(\mathbf{y})P_\varepsilon^{(k)}(\mathbf{x}) \\
&-\sum_{\substack{l\neq j\\ 1\le l\le N}}(\alpha_\varepsilon^{(j)})^{-1}G(\mathbf{O}^{(l)},\mathbf{O}^{(j)})P_\varepsilon^{(l)}(\mathbf{y})P_\varepsilon^{(k)}(\mathbf{x})\Big\} \\
&+O(\max\{|\mathbf{x}-\mathbf{O}^{(m)}|, |\mathbf{y}-\mathbf{O}^{(m)}|\})\,. \qquad (88)
\end{aligned}
$$

Simplifying the second summand in (88), we have

$$\begin{aligned} G_\varepsilon(\mathbf{x},\mathbf{y}) &= g^{(m)}(\boldsymbol{\xi}_m,\boldsymbol{\eta}_m)+(\alpha_\varepsilon^{(m)})^{-1}\zeta^{(m)}(\boldsymbol{\eta}_m)\zeta^{(m)}(\boldsymbol{\xi}_m) \\ &+\sum_{\substack{j\neq m\\ 1\le j\le N}}(\alpha_\varepsilon^{(j)})^{-1}G(\mathbf{y},\mathbf{O}^{(j)})G(\mathbf{x},\mathbf{O}^{(j)}) \\ &+\sum_{j=1}^{N}\sum_{\substack{k\neq j\\ 1\le k\le N}}G(\mathbf{O}^{(k)},\mathbf{O}^{(j)})\Big\{P^{(j)}(\mathbf{y})P^{(k)}(\mathbf{x}) \\ &-\sum_{\substack{l\neq j\\ 1\le l\le N}}(\alpha_\varepsilon^{(j)})^{-1}G(\mathbf{O}^{(l)},\mathbf{O}^{(j)})P^{(l)}(\mathbf{y})P^{(k)}(\mathbf{x})\Big\} \\ &+O(\max\{|\mathbf{x}-\mathbf{O}^{(m)}|,|\mathbf{y}-\mathbf{O}^{(m)}|\})\,, \end{aligned} \tag{89}$$

and since $P_\varepsilon^{(j)}(\mathbf{x})$ is $O((\log\varepsilon)^{-1})$ for $j\neq m$, we arrive at (78). □

5 Asymptotic formulae versus numerical solution

In this section, for the case of when Ω_ε is a planar circular domain with several circular inclusions, we shall compare the asymptotic formula for the regular part $\mathcal{H}_\varepsilon$ of the function G_ε for the operator $-\Delta$, with a solution produced by the method of finite elements in FEMLAB.

The aim of this section is to illustrate through two examples that

i) that the asymptotic formulae can produce a solution to the problem, even when the finite element package cannot, and

ii) that we are able to take the inclusions in our example configurations to be rather large (by increasing ε) and still obtain a good accuracy by the asymptotic formulae.

5.1 Domain and the asymptotic approximation

Let $\Omega\subset\mathbb{R}^2$ be a disk of radius R and let $\mathbf{O}^{(1)},\dots,\mathbf{O}^{(N)}$ be interior points of Ω. We introduce the sets $\omega_\varepsilon^{(j)}$ as disks of positive harmonic capacity in $\mathbb{R}^2$ each with centres $\mathbf{O}^{(j)}$ and small radii $\rho^{(j)}$ for $j=1,\dots,N$, and we have the set $\Omega_\varepsilon = \Omega \setminus \bigcup \omega_\varepsilon^{(j)}$. The function $\mathcal{H}_\varepsilon$ is a solution of the problem

$$\begin{aligned}
\Delta_{\mathbf{x}} \mathcal{H}_\varepsilon(\mathbf{x}, \mathbf{y}) &= 0 , \quad \mathbf{x}, \mathbf{y} \in \Omega_\varepsilon , & (90)\\
\mathcal{H}_\varepsilon(\mathbf{x}, \mathbf{y}) &= -(2\pi)^{-1} \log |\mathbf{x} - \mathbf{y}| , \quad \mathbf{x} \in \partial\Omega_\varepsilon, \mathbf{y} \in \Omega_\varepsilon . & (91)
\end{aligned}$$

The regular part $\mathcal{H}_\varepsilon$ of Green's function G_ε for $-\Delta$ in the domain Ω_ε is given by

$$\begin{aligned}
\mathcal{H}_\varepsilon(\mathbf{x}, \mathbf{y}) &= H(\mathbf{x}, \mathbf{y}) - \sum_{j=1}^{N} g^{(j)}(\mathbf{x} - \mathbf{O}^{(j)}, \mathbf{y} - \mathbf{O}^{(j)}) \\
&\quad -(2\pi)^{-1} N \log |\mathbf{x} - \mathbf{y}| - \sum_{j=1}^{N} \big\{ \alpha^{(j)} \mathcal{P}_\varepsilon^{(j)}(\mathbf{y}) \mathcal{P}_\varepsilon^{(j)}(\mathbf{x}) \\
&\quad -(2\pi)^{-1} \log(\rho^{(j)} (|\mathbf{x} - \mathbf{O}^{(j)}||\mathbf{y} - \mathbf{O}^{(j)}|)^{-1}) \big\} \\
&\quad + \sum_{j=1}^{N} \sum_{\substack{k \neq j \\ 1 \leq k \leq N}} G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)}) \mathcal{P}_\varepsilon^{(k)}(\mathbf{y}) \mathcal{P}_\varepsilon^{(j)}(\mathbf{x}) + O(\varepsilon) , \qquad (92)
\end{aligned}$$

which is uniform with respect to $(\mathbf{x}, \mathbf{y}) \in \Omega_\varepsilon \times \Omega_\varepsilon$. We will use the leading order part of this approximation for our calculations.

Here $\varepsilon = m/d$ is the small parameter, with m being the maximum radius of all the disks $\omega_\varepsilon^{(j)}$ and

$$d = \min\{ \min_{1 \leq j \leq N} \{ \text{dist}(\mathbf{O}^{(j)}, \partial\Omega) \}, \min_{1 \leq i, k \leq N} \{ \text{dist}(\mathbf{O}^{(i)}, \mathbf{O}^{(k)}) \} \} , \qquad (93)$$

the function H is the regular part of Green's function G for the domain Ω

$$H(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi} \log \left(\frac{R}{|\mathbf{y}||\mathbf{x} - \bar{\mathbf{y}}|} \right) , \quad \bar{\mathbf{y}} = \frac{R^2}{|\mathbf{y}|^2} \mathbf{y} ,$$

$g^{(j)}$ is the Green's function for the set $\omega_\varepsilon^{(j)}$, $j = 1, \ldots, N$, given by

$$g^{(j)}(\mathbf{x} - \mathbf{O}^{(j)}, \mathbf{y} - \mathbf{O}^{(j)}) = \frac{1}{2\pi} \log \left(\frac{|\mathbf{y} - \mathbf{O}^{(j)}||\mathbf{x} - \mathbf{O}^{(j)} - \frac{(\rho^{(j)})^2}{|\mathbf{y} - \mathbf{O}^{(j)}|^2} (\mathbf{y} - \mathbf{O}^{(j)})|}{\rho^{(j)} |\mathbf{x} - \mathbf{y}|} \right) .$$

The function $\mathcal{P}_\varepsilon^{(j)}$ is the leading part of the approximation of the function $P_\varepsilon^{(j)}$, $j = 1, \ldots, N$ which is a solution of

$$\begin{aligned}
\Delta_{\mathbf{x}} P_\varepsilon^{(j)}(\mathbf{x}) &= 0 , \quad \mathbf{x} \in \Omega_\varepsilon , & (94)\\
P_\varepsilon^{(j)}(\mathbf{x}) &= 0 , \quad \mathbf{x} \in \partial\Omega , & (95)\\
P_\varepsilon^{(j)}(\mathbf{x}) &= \delta_{kj} , \quad \mathbf{x} \in \partial\omega_\varepsilon^{(k)} , k = 1, \ldots, N . & (96)
\end{aligned}$$

Let $\mathcal{P}_\varepsilon = \{\mathcal{P}_\varepsilon^{(j)}\}_{j=1}^N$, then the entries $\mathcal{P}_\varepsilon^{(j)}$ are obtained from

$$\mathcal{P}_\varepsilon(\mathbf{x}) = \left(\operatorname*{diag}_{1\le j\le N}\{\alpha^{(j)}\} - \mathfrak{M}\right)^{-1}\mathcal{S}(\mathbf{x}) \ , \tag{97}$$

where $\alpha^{(j)} = (2\pi)^{-1}\log\rho^{(j)} + H(\mathbf{O}^{(j)}, \mathbf{O}^{(j)})$, $\mathfrak{M} = ((1-\delta_{kj})G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)}))_{j,k=1}^N$, with

$$G(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi}\log|\mathbf{x}-\mathbf{y}| - H(\mathbf{x}, \mathbf{y}) \ ,$$

and $\mathcal{S} = \{\mathcal{S}^{(j)}\}_{j=1}^N$ with entries being given by $\mathcal{S}^{(j)}(\mathbf{x}) = -G(\mathbf{x}, \mathbf{O}^{(j)})$.

The formula (92) can be written via solutions of model problems in domains independent of the small parameter.

Let the sets $\omega^{(j)} = \{\varepsilon^{-1}(\mathbf{x}-\mathbf{O}^{(j)}) : \mathbf{x} \in \omega_\varepsilon^{(j)}\}$, $j = 1, \dots, N$ with radii $r^{(j)} = \varepsilon^{-1}\rho^{(j)}$, and denote there complements by $C\bar{\omega}^{(j)} = \mathbb{R}^2 \setminus \omega^{(j)}$, $j = 1, \dots, N$.

We will assume that all of $\omega^{(j)}$ contain the origin and that the maximum distance between the $\mathbf{O}$ and $\partial\omega^{(j)}$ is equal to d.

In the following we use the scaled variables $\boldsymbol{\xi}_j = \varepsilon^{-1}(\mathbf{x}-\mathbf{O}^{(j)})$ and $\boldsymbol{\eta}_j = \varepsilon^{-1}(\mathbf{y}-\mathbf{O}^{(j)})$. The Green's functions for the sets $\omega^{(j)}$, $j = 1, \dots, N$ are given by

$$g^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) = \frac{1}{2\pi}\log\left(\frac{|\boldsymbol{\eta}_j||\boldsymbol{\xi}_j - \bar{\boldsymbol{\eta}}_j|}{r^{(j)}|\boldsymbol{\xi}_j - \boldsymbol{\eta}_j|}\right) \ , \quad \bar{\boldsymbol{\eta}}_j = \frac{(r^{(j)})^2}{|\boldsymbol{\eta}_j|^2}\boldsymbol{\eta}_j \ , \tag{98}$$

We introduce the functions $\zeta^{(j)}$ by

$$\zeta^{(j)}(\boldsymbol{\eta}_j) = \lim_{|\boldsymbol{\xi}_j|\to\infty} g^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) \ , \tag{99}$$

and the constants

$$\zeta_\infty^{(j)} = \lim_{|\boldsymbol{\eta}_j|\to\infty}\{\zeta^{(j)}(\boldsymbol{\eta}_j) - (2\pi)^{-1}\log|\boldsymbol{\eta}_j|\} \ , \tag{100}$$

for $j = 1, \dots, N$. For the domain Ω_ε described above

$$\zeta^{(j)}(\boldsymbol{\eta}_j) = \frac{1}{2\pi}\log\left(\frac{|\boldsymbol{\eta}_j|}{r^{(j)}}\right) \ , \quad \zeta_\infty^{(j)} = -\frac{1}{2\pi}\log r^{(j)} \ . \tag{101}$$

We may then rewrite (92), incorporating the small parameter ε with the use of (98), (99) and (101) as follows

$$
\begin{aligned}
\mathcal{H}_\varepsilon(\mathbf{x},\mathbf{y}) &= H(\mathbf{x},\mathbf{y}) - \sum_{j=1}^{N} g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) - (2\pi)^{-1} N \log(\varepsilon^{-1}|\mathbf{x}-\mathbf{y}|) \\
&\quad - \sum_{j=1}^{N} \{\alpha_\varepsilon^{(j)} \mathcal{P}_\varepsilon^{(j)}(\mathbf{y}) \mathcal{P}_\varepsilon^{(j)}(\mathbf{x}) - \zeta^{(j)}(\boldsymbol{\xi}_j) - \zeta^{(j)}(\boldsymbol{\eta}_j) + \zeta_\infty^{(j)}\} \\
&\quad + \sum_{j=1}^{N} \sum_{\substack{k\neq j \\ 1\le k\le N}} G(\mathbf{O}^{(k)},\mathbf{O}^{(j)}) \mathcal{P}_\varepsilon^{(k)}(\mathbf{y}) \mathcal{P}_\varepsilon^{(j)}(\mathbf{x}) + O(\varepsilon)\;, \qquad (102)
\end{aligned}
$$

where $\alpha_\varepsilon^{(j)} = (2\pi)^{-1}\log\varepsilon + (2\pi)^{-1}\log r^{(j)} + H(\mathbf{O}^{(j)},\mathbf{O}^{(j)})$.

5.2 Example: A configuration with a large number of small inclusions

For our first illustrative example, we shall plot the regular part $\mathcal{H}_\varepsilon$ of Green's function G_ε.

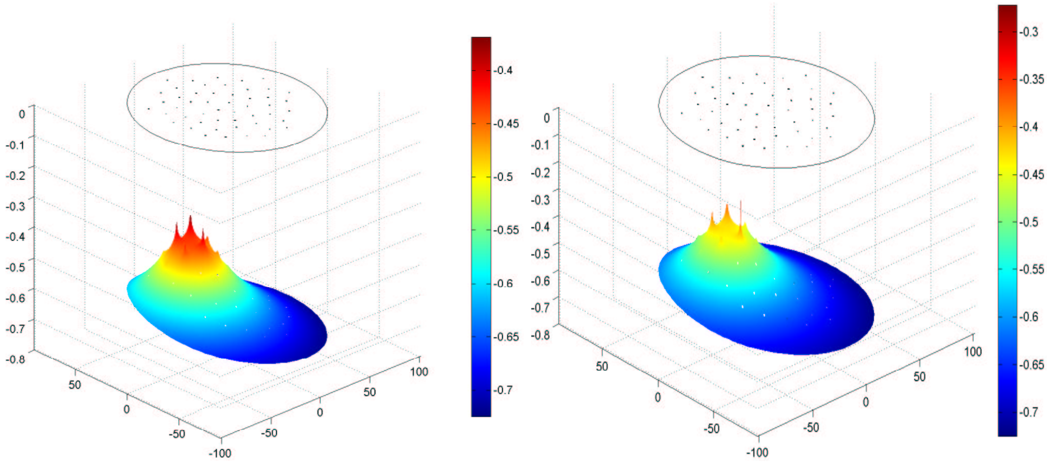
We produced the surface plot of the asymptotic solution for $\mathcal{H}_\varepsilon$, on a mesh consisting of 752448 elements, (see Fig 4). On this mesh, FEMLAB was unable to produce an accurate numerical solution, but the asymptotic formula is still efficient for this case.

The numerical settings are as follows. Let Ω be the disk of radius $R = 70$, centered at the origin. We consider the situation when we have $N = 50$ small disks, whose radii in scaled coordinates do not exceed 10.0449, and our small parameter $\varepsilon = 0.0498$. The location of the point force is given by $\mathbf{y} = (-20, 15)$.

For a mesh containing 188112 elements, we produced a surface plot of the asymptotic formula for $\mathcal{H}_\varepsilon$ given in (102) and the numerical solution given in FEMLAB by the method of finite elements, and the corresponding diagrams are shown in Fig 2 a), b).

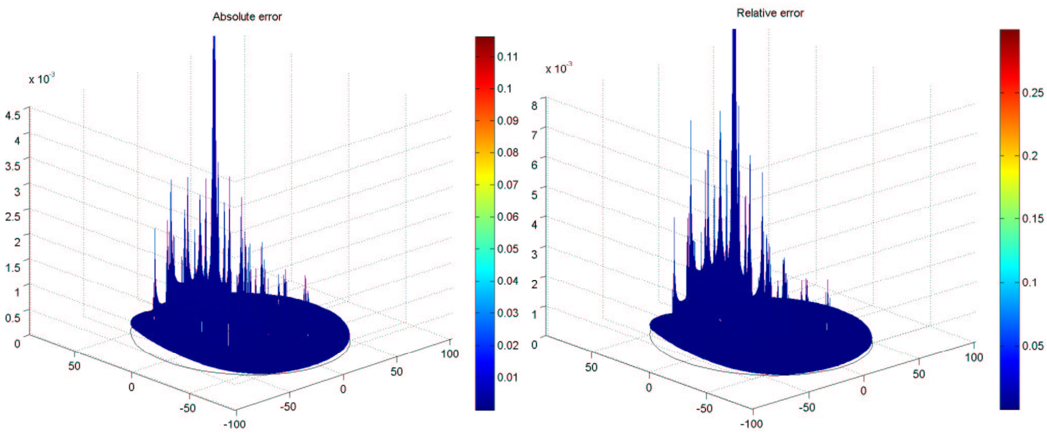
We compared both the asymptotic representation for the regular part of G_ε and the numerical solution produced in FEMLAB on this mesh, by taking the absolute difference between the two (see Fig 3 a)) and then the relative error (see Fig 3 b)). From both of these figures it can be seen that the asymptotic formula gives a good approximation to the numerical solution produced in FEMLAB.

The critical case when FEMLAB failed but the asymptotic formula still produced an accurate solution is shown in Fig. 4.



a) b)

Figure 2: a) Numerical solution produced in FEMLAB on a mesh containing 188112 elements, b) Computation based on the asymptotic formula for $\mathcal{H}_\varepsilon$, when $\varepsilon = 0.0498$.



a) b)

Figure 3: a) Absolute error and b) relative error between numerical solution and the computations produced by the asymptotic formula for $\mathcal{H}_\varepsilon$, when $\varepsilon = 0.0498$ and the mesh contains 188112 elements. All the spikes occur on the boundaries of the inclusions. Maximum absolute error is 0.1162, maximum relative error is 0.2995, which is attained on the boundary of the inclusion with centre (-20, 4), near the point (-20, 15) where the force is applied.

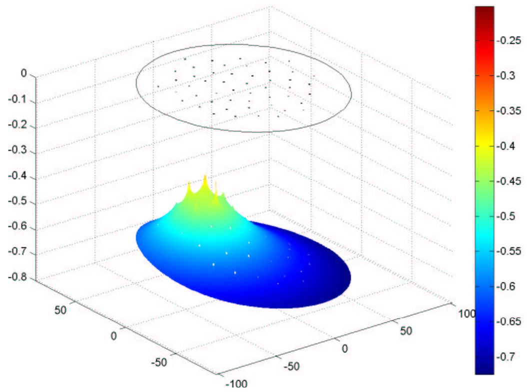


Figure 4: The computation based on the asymptotic formula for the regular part $\mathcal{H}_\varepsilon$ of Green's function on the refined mesh, when $\varepsilon = 0.0498$ and the mesh contains 752448 elements.

5.3 Example: A configuration with inclusions of relatively large size

In this example, we shall once again take the asymptotic formulae for the regular part $\mathcal{H}_\varepsilon$ of the function G_ε and compare this with numerical solutions produced in FEMLAB, for a configuration with few inclusions, and we shall experiment with our parameter ε. We show that we are able to consider a configuration where the inclusions are rather large (by increasing ε) and our asymptotic formula for $\mathcal{H}_\varepsilon$ still gives a good approximation to the numerical solution.

Let Ω now be a disk of radius 150, and we consider the case when we have 5 inclusions $\omega_\varepsilon^{(j)}$, $j = 1, \ldots, 5$, with centres $\mathbf{O}^{(1)} = (44, 66)$, $\mathbf{O}^{(2)} = (-90, 34)$, $\mathbf{O}^{(3)} = (-36, -68)$, $\mathbf{O}^{(4)} = (68, -26)$, $\mathbf{O}^{(5)} = (-14, 0)$, and whose radii in scaled coordinates do not exceed 53.7919. The position of the point force is $\mathbf{y} = (-25, 70)$.

In Table 1, we present data showing how the error between the numerical solution given in FEMLAB and the asymptotic formula for the regular part of Green's function $\mathcal{H}_\varepsilon$ changes as we decrease ε. Here m denotes the maximum radius of the inclusions and A_{max} and R_{max} are absolute and relative error, respectively.

We also have for the situation when $\varepsilon = 0.7436$ the surface plot of the asymptotic formula for the regular part of Green's function and the relative error between the numerical solution and the asymptotic formula; we note that inclusions are rather large in this case (see Fig 5 a) and b)). It can be seen from Fig 5 b) that although the maximum relative error is larger near where the point force is applied ($R_{max} = 0.1991$), the asymptotic formula

m	ε	A_{max}	R_{max}
40	0.7436	0.1219	0.1991
36	0.6692	0.09741	0.157
32	0.5949	0.07637	0.1216
28	0.5205	0.05845	0.09204
24	0.4462	0.04335	0.06752
20	0.3718	0.0308	0.04749
16	0.2974	0.0206	0.03156
12	0.2231	0.01298	0.02
8	0.1487	0.007266	0.0111
4	0.0744	0.001395	0.004503
2	0.0372	0.0006608	0.001991
1	0.0186	0.002993	0.0009269
0.5	0.0093	0.0003156	0.0004448
0.25	0.0046	0.0001515	0.0002171

Table 1: Maximum absolute and relative error corresponding to various values of ε.

still gives a good match with the numerical solution everywhere else.

The plot of ε against R_{max} on a logarithmic scale is shown in Fig. 6. It can be seen from this that for small ε the graph is appears to be linear and from this we can conclude the numerical evaluation of the relative error R_{max} is consistent with the theoretical prediction of formula (102).

6 Green's tensor for the Lamé operator in two dimensional elasticity

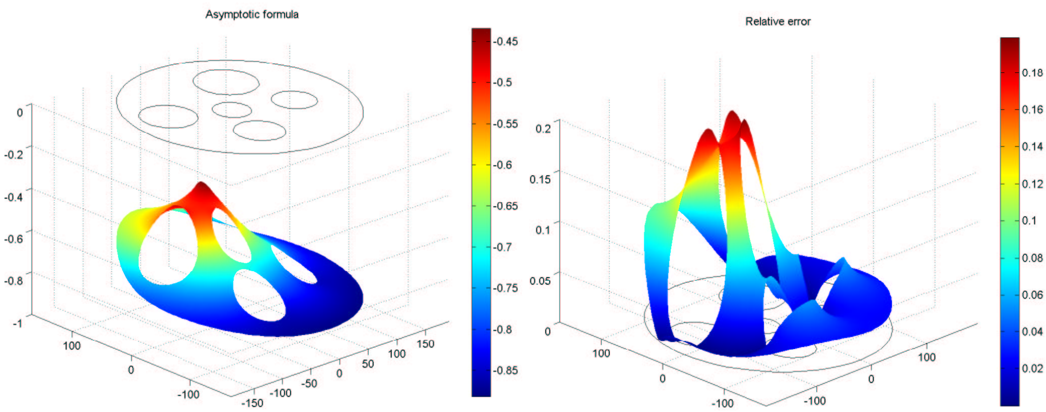
In the subsequent sections we shall study Green's tensor for the Lamé operator in $\Omega_\varepsilon \subset \mathbb{R}^n$, $n = 2, 3$ which will be denoted by G_ε. The tensor G_ε is a solution of

$$\mu\Delta_{\mathbf{x}}G_\varepsilon(\mathbf{x},\mathbf{y}) + (\lambda+\mu)\nabla_{\mathbf{x}}(\nabla_{\mathbf{x}} \cdot G_\varepsilon(\mathbf{x},\mathbf{y})) + \delta(\mathbf{x}-\mathbf{y})I_n = 0I_n \ , \quad \mathbf{x},\mathbf{y} \in \Omega_\varepsilon \ , \tag{103}$$

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = 0I_n \ , \quad \mathbf{x} \in \partial\Omega_\varepsilon, \mathbf{y} \in \Omega_\varepsilon \ , \tag{104}$$

where I_n is the $n \times n$ identity matrix. An important property of this tensor is the following symmetry relation

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = G_\varepsilon^T(\mathbf{y},\mathbf{x}) \ . \tag{105}$$



a) b)

Figure 5: a) Computations produced by the asymptotic formula for $\mathcal{H}_\varepsilon$, b) The relative error between the numerical solution and the asymptotic formula for the case $\varepsilon = 0.7436$.

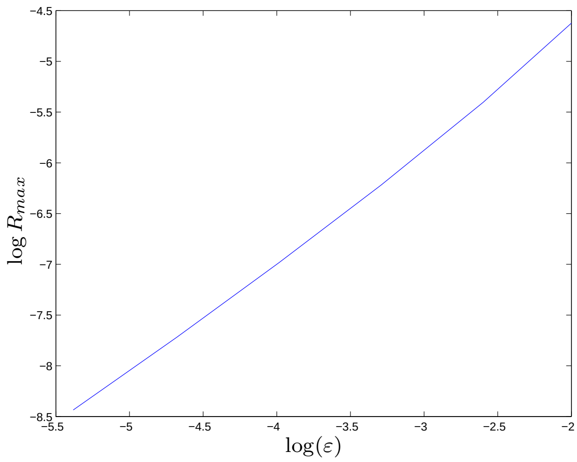


Figure 6: Plot of $\log(\varepsilon)$ against $\log R_{max}$.

We shall also introduce the notation for the Lamé operator

$$L(\partial_{\mathbf{x}}) := \mu\Delta_{\mathbf{x}} + (\lambda+\mu)\nabla_{\mathbf{x}}(\nabla_{\mathbf{x}}\cdot\,)\ ,$$

where $\partial_{\mathbf{x}} = \partial/\partial\mathbf{x}$.

Let $\mathbf{u}$ be the displacement vector which satisfies the Dirichlet boundary value problem in the domain $\Omega_\varepsilon \subset \mathbb{R}^n$, $n = 2, 3$

$$L\left(\partial_{\mathbf{x}}\right)\mathbf{u}(\mathbf{x}) = \mathbf{O}\ , \quad \mathbf{x} \in \Omega_\varepsilon\ , \tag{106}$$

$$\mathbf{u}(\mathbf{x}) \;=\; \boldsymbol{\psi}(\mathbf{x})\ , \quad \mathbf{x} \in \partial\Omega\ , \tag{107}$$

$$\mathbf{u}(\mathbf{x}) \;=\; \boldsymbol{\varphi}_j(\varepsilon^{-1}\mathbf{x})\ , \quad \mathbf{x} \in \partial\omega_\varepsilon^{(j)}\ , 1 \le j \le N\ , \tag{108}$$

where $\mathbf{O}$ is the zero vector, and we assume that $\boldsymbol{\varphi}_j$ and $\boldsymbol{\psi}$ are continuous vector functions.

We have the following Lemma, whose analog for the case of a single inclusion was proved in [10].

Lemma 5 *There exists a unique solution* $\mathbf{u} \in C(\bar{\Omega}_\varepsilon)$ *of problem* (106)$-$(108) *which satisfies the estimate*

$$\max_{\bar{\Omega}_\varepsilon} |\mathbf{u}(\mathbf{x})| \le \text{const}\,\max\{\max_{1\le j\le N}\{\|\boldsymbol{\varphi}_j\|_{C(\partial\omega_\varepsilon^{(j)})}\}\ , \|\boldsymbol{\psi}\|_{C(\partial\Omega)}\}\ . \tag{109}$$

The preceding result will be used to obtain the estimates for the remainders produced by our asymptotic approximations.

6.1 Green's Matrix for a 2-dimensional domain with several small inclusions

In this section, we shall consider the uniform approximation of the tensor $G_\varepsilon(\mathbf{x},\mathbf{y})$ for the case of a planar domain with multiple small inclusions ($n = 2$), formulated in section 6. We once again introduce model domains and governing equations needed for the study related to this case.

6.2 Green's kernels for model domains in two dimensions

Let $G(\mathbf{x},\mathbf{y}) = [G^{(1)}(\mathbf{x},\mathbf{y}), G^{(2)}(\mathbf{x},\mathbf{y})]$ and $g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) = [g^{(j,1)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j), g^{(j,2)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)]$ now denote Green's tensors for the Lamé operator in the domain Ω and $C\bar{\omega}^{(j)} = \mathbb{R}^2\backslash\bar{\omega}^{(j)}$, $j = 1, \dots, N$, respectively. The tensor G is a solution the following problem

$$L(\partial_{\mathbf{x}})G(\mathbf{x},\mathbf{y}) + \delta(\mathbf{x}-\mathbf{y})I_2 = 0I_2\ , \quad \mathbf{x},\mathbf{y} \in \Omega\ , \tag{110}$$

$$G(\mathbf{x},\mathbf{y}) = 0I_2\ , \quad \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega\ , \tag{111}$$

and the tensors $g^{(j)}$ solve

$$L(\partial_{\boldsymbol{\xi}_j})g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) + \delta(\boldsymbol{\xi}_j - \boldsymbol{\eta}_j)I_2 = 0I_2\ , \quad \boldsymbol{\xi}_j,\boldsymbol{\eta}_j \in C\bar{\omega}^{(j)}\ , \tag{112}$$

$$g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) = 0I_2\ , \quad \boldsymbol{\xi}_j \in \partial C\bar{\omega}^{(j)}, \boldsymbol{\eta}_j \in C\bar{\omega}^{(j)}\ , \tag{113}$$

$$|g^{(j,k)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)| \text{ is bounded as } |\boldsymbol{\xi}_j| \to \infty\ , \boldsymbol{\eta}_j \in C\bar{\omega}^{(j)} \text{ for } k = 1,2\ . \tag{114}$$

We represent $G(\mathbf{x},\mathbf{y})$ as

$$G(\mathbf{x},\mathbf{y}) = \gamma(\mathbf{x},\mathbf{y}) - H(\mathbf{x},\mathbf{y})\ , \tag{115}$$

and $g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)$ for $j = 1,\dots,N$ as

$$g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) = \gamma(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) - h^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)\ , \tag{116}$$

where H and $h^{(j)}$ are the regular parts of G and $g^{(j)}$, respectively, and $\gamma(\mathbf{x},\mathbf{y}) = [\gamma_{ij}(\mathbf{x},\mathbf{y})]_{i,j=1}^2$, is the fundamental solution of the Lamé operator in two dimensions with components

$$\begin{aligned}\gamma_{ij}(\mathbf{x},\mathbf{y}) \;&=\; (\lambda+3\mu)(4\pi\mu(\lambda+2\mu))^{-1}(-\log|\mathbf{x}-\mathbf{y}|\delta_{ij} \\ &\quad +(\lambda+\mu)(\lambda+3\mu)^{-1}(x_i-y_i)(x_j-y_j)|\mathbf{x}-\mathbf{y}|^{-2})\ ,\end{aligned} \tag{117}$$

for $i,j = 1,2$. We introduce the tensor $\zeta^{(j)}$ as

$$\zeta^{(j)}(\boldsymbol{\eta}_j) = \lim_{|\boldsymbol{\xi}_j|\to\infty} g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)\ , \tag{118}$$

and the constant matrix

$$\zeta^{(\infty,j)} = \lim_{|\boldsymbol{\eta}_j|\to\infty} \{\zeta^{(j)}(\boldsymbol{\eta}_j) + \gamma(\boldsymbol{\eta}_j,\mathbf{O})\}\ , \tag{119}$$

for $j = 1,\dots,N$.

In [10], it was proved that the matrices $\zeta^{(j)}$, $\zeta^{(\infty,j)}$, $1 \le j \le N$, where symmetric.

6.3 Auxiliary matrix functions

6.3.1 An estimate for the regular part $h^{(j)}$ of Green's tensor for the unbounded domain

Here we state a result concerning an asymptotic expansion of the regular part $h^{(j)}$ of Green's tensor $g^{(j)}$, which is consequence of Lemma 2 presented in [7], (p. 78).

For the proof of the following Lemmas, we refer to [7].

Lemma 6 *Let $|\boldsymbol{\xi}_j| > 2$. Then the regular part $h^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j)$ of Green's matrix $g^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j)$, in $C\bar{\omega}^{(j)}$ admits the asymptotic representation*

$$h^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) = \gamma(\boldsymbol{\xi}_j, \mathbf{O}) - \zeta^{(j)}(\boldsymbol{\eta}_j) + O(|\boldsymbol{\xi}_j|^{-1}) , \tag{120}$$

for $j = 1, \dots, N$.

We also have the following asymptotic representation of the matrix function $\zeta^{(j)}$

Lemma 7 *For $|\boldsymbol{\xi}_j| > 2$, the following representation for $\zeta^{(j)}$ holds*

$$\zeta^{(j)}(\boldsymbol{\xi}_j) = -\gamma(\boldsymbol{\xi}_j, \mathbf{O}) + \zeta^{(\infty,j)} + O(|\boldsymbol{\xi}_j|^{-1}) , \tag{121}$$

for $j = 1, \dots, N$.

6.3.2 The elastic capacitary potential

Let $P_\varepsilon^{(j)}(\mathbf{x})$ be elastic capacitary potential corresponding to the j^{th} inclusion. The matrix $P_\varepsilon^{(j)}(\mathbf{x})$ is defined as a solution of

$$L(\partial_{\mathbf{x}})P_\varepsilon^{(j)}(\mathbf{x}) = 0I_2, \quad \mathbf{x} \in \Omega_\varepsilon , \tag{122}$$

$$P_\varepsilon^{(j)}(\mathbf{x}) = 0I_2 , \quad \mathbf{x} \in \partial\Omega , \tag{123}$$

$$P_\varepsilon^{(j)}(\mathbf{x}) = I_2 , \quad \mathbf{x} \in \partial\omega_\varepsilon^{(j)} , \tag{124}$$

$$P_\varepsilon^{(j)}(\mathbf{x}) = 0I_2 , \quad \mathbf{x} \in \partial\omega_\varepsilon^{(k)} , 1 \le k \le N , k \ne j . \tag{125}$$

Given the above boundary value problem, we now consider the approximation of the matrix $P_\varepsilon^{(j)}(\mathbf{x})$.

Lemma 8 *The leading order part $\mathcal{P}_\varepsilon^{(j)}$ of the asymptotic approximation of $P_\varepsilon^{(j)}(\mathbf{x})$ is a solution of the following system of equations*

$$\begin{aligned} \mathcal{P}_\varepsilon^{(j)}(\mathbf{x}) \;=\; & \Big(G(\mathbf{x}, \mathbf{O}^{(j)}) - \zeta^{(j)}(\boldsymbol{\xi}_j) - \gamma(\boldsymbol{\xi}_j, \mathbf{O}) + \zeta^{(\infty,j)} \\ & - \sum_{\substack{k \ne j \\ 1 \le k \le N}} \mathcal{P}_\varepsilon^{(k)}(\mathbf{x}) G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)}) \Big) D^{(j)} , \end{aligned} \tag{126}$$

where $D^{(j)} = [D_{ik}^{(j)}]_{i,k=1}^2$ has entries given by

$$D_{11}^{(j)} = -(K_1^{(j)})^{-1}(K_2 \log\varepsilon - \zeta_{22}^{(\infty,j)} + H_{22}(\mathbf{O}^{(j)}, \mathbf{O}^{(j)})) , \tag{127}$$

$$D_{12}^{(j)} = -(K_1^{(j)})^{-1}(\zeta_{12}^{(\infty,j)} - H_{12}(\mathbf{O}^{(j)}, \mathbf{O}^{(j)})) , \tag{128}$$

$$D_{21}^{(j)} = -(K_1^{(j)})^{-1}(\zeta_{21}^{(\infty,j)} - H_{21}(\mathbf{O}^{(j)}, \mathbf{O}^{(j)})) , \tag{129}$$

$$D_{22}^{(j)} = -(K_1^{(j)})^{-1}(K_2 \log\varepsilon - \zeta_{11}^{(\infty,j)} + H_{11}(\mathbf{O}^{(j)}, \mathbf{O}^{(j)})) , \tag{130}$$

and

$$\begin{aligned} K_1^{(j)} &= \Big(K_2 \log \varepsilon - \zeta_{11}^{(\infty,j)} + H_{11}(\mathbf{O}^{(j)}, \mathbf{O}^{(j)})\Big) \\ &\quad \times \Big(K_2 \log \varepsilon - \zeta_{22}^{(\infty,j)} + H_{22}(\mathbf{O}^{(j)}, \mathbf{O}^{(j)})\Big) \\ &\quad -(H_{12}(\mathbf{O}^{(j)}, \mathbf{O}^{(j)}) - \zeta_{12}^{(\infty,j)})(H_{21}(\mathbf{O}^{(j)}, \mathbf{O}^{(j)}) - \zeta_{21}^{(\infty,j)}) \ , \quad (131) \end{aligned}$$

$$K_2 = \frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)} \ , \tag{132}$$

for $j = 1, \dots, N$.

Proof. We represent $P_\varepsilon^{(j)}(\mathbf{x})$ in the form

$$P_\varepsilon^{(j)}(\mathbf{x}) = (G(\mathbf{x}, \mathbf{O}^{(j)}) - \zeta^{(j)}(\boldsymbol{\xi}_j) - \gamma(\boldsymbol{\xi}_j, \mathbf{O}) + \zeta^{(\infty,j)})D^{(j)} + R_\varepsilon^{(j)}(\mathbf{x}) \ , 1 \le j \le N \ , \tag{133}$$

where the matrix $R_\varepsilon^{(j)}(\mathbf{x})$ satisfies

$$L(\partial_{\mathbf{x}})R_\varepsilon^{(j)}(\mathbf{x}) = 0I_2, \quad \mathbf{x} \in \Omega_\varepsilon \ , \tag{134}$$

$$R_\varepsilon^{(j)}(\mathbf{x}) = (\zeta^{(j)}(\boldsymbol{\xi}_j) + \gamma(\boldsymbol{\xi}_j, \mathbf{O}) - \zeta^{(\infty,j)})D^{(j)} \ , \quad \mathbf{x} \in \partial\Omega \ , \tag{135}$$

$$R_\varepsilon^{(j)}(\mathbf{x}) = I_2 - \big(-K_2 \log \varepsilon I_2 - H(\mathbf{x}, \mathbf{O}^{(j)}) + \zeta^{(\infty,j)}\big) D^{(j)} \ , \quad \mathbf{x} \in \partial\omega_\varepsilon^{(j)} \ , \tag{136}$$

$$\begin{aligned} R_\varepsilon^{(j)}(\mathbf{x}) &= -(G(\mathbf{x}, \mathbf{O}^{(j)}) - \zeta^{(j)}(\boldsymbol{\xi}_j) - \gamma(\boldsymbol{\xi}_j, \mathbf{O}) + \zeta^{(\infty,j)})D^{(j)} \ , \\ &\quad \mathbf{x} \in \partial\omega_\varepsilon^{(k)} \ 1 \le k \le N \ , k \ne j \ . \end{aligned} \tag{137}$$

The boundary condition (136) is equivalent to

$$R_\varepsilon^{(j)}(\mathbf{x}) = (H(\mathbf{x}, \mathbf{O}^{(j)}) - H(\mathbf{O}^{(j)}, \mathbf{O}^{(j)}))D^{(j)} \ , \quad \mathbf{x} \in \partial\omega_\varepsilon^{(j)} \ , \tag{138}$$

where $D^{(j)} = O((\log \varepsilon)^{-1})$, so $R_\varepsilon^{(j)}(\mathbf{x}) = O(\varepsilon(\log \varepsilon)^{-1})$ for $\mathbf{x} \in \partial\omega_\varepsilon^{(j)}$.

By Lemma 7

$$\zeta^{(j)}(\boldsymbol{\xi}_j) + \gamma(\boldsymbol{\xi}_j, \mathbf{O}) - \zeta^{(\infty,j)} = O(\varepsilon), \quad \text{for } \mathbf{x} \in \partial\Omega \ . \tag{139}$$

Then in (135), we have that $R_\varepsilon^{(j)}(\mathbf{x}) = O(\varepsilon(\log \varepsilon)^{-1})$ for $\mathbf{x} \in \partial\Omega$.

Next, using Lemma 7 and the fact that $G(\mathbf{x}, \mathbf{O}^{(j)})$ is smooth for $\mathbf{x} \in \Omega_\varepsilon$, we have in (137)

$$R_\varepsilon^{(j)}(\mathbf{x}) = -G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)})D^{(j)} + O(\varepsilon(\log \varepsilon)^{-1}) \ , \tag{140}$$

for $\mathbf{x} \in \partial\omega_\varepsilon^{(k)}, 1 \le k \le N, k \ne j$.

Then we may write $R_\varepsilon^{(j)}(\mathbf{x})$, using the elastic capacitary potential for the individual inclusions, as

$$R_\varepsilon^{(j)}(\mathbf{x}) = -\sum_{\substack{k\neq j\\ 1\leq k\leq N}} P_\varepsilon^{(k)}(\mathbf{x})G(\mathbf{O}^{(k)},\mathbf{O}^{(j)})D^{(j)} + p^{(j)}(\mathbf{x})\ . \tag{141}$$

Combining (133) and (141) we arrive at

$$\begin{aligned} P_\varepsilon^{(j)}(\mathbf{x}) &= \Big(G(\mathbf{x},\mathbf{O}^{(j)}) - \zeta^{(j)}(\boldsymbol{\xi}_j) - \gamma(\boldsymbol{\xi}_j,\mathbf{O}) + \zeta^{(\infty,j)} \\ &\quad - \sum_{\substack{k\neq j\\ 1\leq k\leq N}} P_\varepsilon^{(k)}(\mathbf{x})G(\mathbf{O}^{(k)},\mathbf{O}^{(j)})\Big)D^{(j)} + p^{(j)}(\mathbf{x})\ . \end{aligned} \tag{142}$$

Here $p^{(j)}(\mathbf{x})$ is a matrix satisfying the homogeneous Lamé equation, and is $O(\varepsilon(\log\varepsilon)^{-1})$ for $\mathbf{x}\in\partial\Omega$ and $\mathbf{x}\in\partial\omega_\varepsilon^{(j)}$, $1\leq j\leq N$. Therefore by Lemma 5, $p^{(j)}(\mathbf{x})$ for $1\leq j\leq N$ is $O(\varepsilon(\log\varepsilon)^{-1})$ uniformly with respect to $\mathbf{x}\in\Omega_\varepsilon$.

The removal of the remainder term in (142), we have the system (126). □

6.4 A uniform asymptotic formula

Now we may approach the approximation of Green's matrix G_ε for a 2-dimensional elastic solid with multiple inclusions.

Theorem 3 *Green's tensor for the Lamé operator in $\Omega_\varepsilon\subset\mathbb{R}^2$ admits the representation*

$$\begin{aligned} G_\varepsilon(\mathbf{x},\mathbf{y}) &= G(\mathbf{x},\mathbf{y}) + \sum_{j=1}^N g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) - N\gamma(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) \\ &\quad + \sum_{j=1}^N \left\{P_\varepsilon^{(j)}(\mathbf{x})A^{(j)}P_\varepsilon^{(j)T}(\mathbf{y}) - \zeta^{(j)}(\boldsymbol{\xi}_j) - \zeta^{(j)}(\boldsymbol{\eta}_j) + \zeta^{(\infty,j)}\right\} \\ &\quad - \sum_{j=1}^N \sum_{\substack{k\neq j\\ 1\leq k\leq N}} P_\varepsilon^{(j)}(\mathbf{x})G(\mathbf{O}^{(j)},\mathbf{O}^{(k)})P_\varepsilon^{(k)T}(\mathbf{y}) + O(\varepsilon)\ , \end{aligned} \tag{143}$$

uniformly with respect to $(\mathbf{x},\mathbf{y})\in\Omega_\varepsilon\times\Omega_\varepsilon$, *where*

$$A^{(j)} = K_2\log\varepsilon I_2 + H(\mathbf{O}^{(j)},\mathbf{O}^{(j)}) - \zeta^{(\infty,j)}\ ,\quad 1\leq j\leq N\ . \tag{144}$$

Proof. Let G_ε be sought in the form

$$G_\varepsilon(\mathbf{x},\mathbf{y}) = \gamma(\mathbf{x},\mathbf{y}) - H_\varepsilon(\mathbf{x},\mathbf{y}) - \sum_{j=1}^{N} h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) \,, \tag{145}$$

where it suffices to seek the approximation of the tensors $H_\varepsilon(\mathbf{x},\mathbf{y})$ and $h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})$, which solve the problems

$$L(\partial_{\mathbf{x}})H_\varepsilon(\mathbf{x},\mathbf{y}) = 0I_2 \,, \quad \mathbf{x},\mathbf{y} \in \Omega_\varepsilon \,, \tag{146}$$

$$H_\varepsilon(\mathbf{x},\mathbf{y}) = \gamma(\mathbf{x},\mathbf{y}) \,, \quad \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega_\varepsilon \,, \tag{147}$$

$$H_\varepsilon(\mathbf{x},\mathbf{y}) = 0I_2 \,, \quad \mathbf{x} \in \partial\omega_\varepsilon^{(j)}, \mathbf{y} \in \Omega_\varepsilon, 1 \le j \le N \,, \tag{148}$$

and

$$L(\partial_{\mathbf{x}})h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = 0I_2 \,, \quad \mathbf{x},\mathbf{y} \in \Omega_\varepsilon \,, \tag{149}$$

$$h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = 0I_2 \,, \quad \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega_\varepsilon \,, \tag{150}$$

$$h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = \gamma(\mathbf{x},\mathbf{y}) \,, \quad \mathbf{x} \in \partial\omega_\varepsilon^{(j)}, \mathbf{y} \in \Omega_\varepsilon \,, \tag{151}$$

$$h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = 0I_2 \,, \quad \mathbf{x} \in \partial\omega_\varepsilon^{(k)}, \mathbf{y} \in \Omega_\varepsilon \,, 1 \le k \le N \,, k \ne j \,. \tag{152}$$

The approximation of $H_\varepsilon(\mathbf{x},\mathbf{y})$. Let $H_\varepsilon(\mathbf{x},\mathbf{y})$ be given by

$$H_\varepsilon(\mathbf{x},\mathbf{y}) = -P_\varepsilon^{(j)}(\mathbf{x})H(\mathbf{O}^{(j)},\mathbf{y}) + H(\mathbf{x},\mathbf{y}) + V(\mathbf{x},\mathbf{y}) \,, \tag{153}$$

where $V(\mathbf{x},\mathbf{y})$ satisfies

$$L(\partial_{\mathbf{x}})V(\mathbf{x},\mathbf{y}) = 0I_2 \,, \quad \mathbf{x},\mathbf{y} \in \Omega_\varepsilon \,, \tag{154}$$

$$V(\mathbf{x},\mathbf{y}) = 0I_2 \,, \quad \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega_\varepsilon \,, \tag{155}$$

$$V(\mathbf{x},\mathbf{y}) = H(\mathbf{O}^{(j)},\mathbf{y}) - H(\mathbf{x},\mathbf{y}) \,, \quad \mathbf{x} \in \partial\omega_\varepsilon^{(j)}, \mathbf{y} \in \Omega_\varepsilon \,, \tag{156}$$

$$V(\mathbf{x},\mathbf{y}) = -H(\mathbf{x},\mathbf{y}) \,, \quad \mathbf{x} \in \partial\omega_\varepsilon^{(k)}, \mathbf{y} \in \Omega_\varepsilon \,, k \ne j \,, 1 \le k \le N \,. \tag{157}$$

Since $\omega_\varepsilon^{(j)}$, $1 \le j \le N$, are small inclusions and H is a smooth tensor in Ω we may expand H about their centres. Namely, for the boundary condition (156) we have

$$V(\mathbf{x},\mathbf{y}) = H(\mathbf{O}^{(j)},\mathbf{y}) - H(\mathbf{x},\mathbf{y}) = O(\varepsilon) \,, \quad \mathbf{x} \in \partial\omega_\varepsilon^{(j)}, \mathbf{y} \in \Omega_\varepsilon \,, \tag{158}$$

and from (157)

$$\begin{aligned} V(\mathbf{x},\mathbf{y}) \; &= \; -H(\mathbf{x},\mathbf{y}) = -H(\mathbf{O}^{(k)},\mathbf{y}) + O(\varepsilon) \,, \\ &\qquad \mathbf{x} \in \partial\omega_\varepsilon^{(k)}, \mathbf{y} \in \Omega_\varepsilon \,, k \ne j \,, 1 \le k \le N \,. \end{aligned} \tag{159}$$

Therefore, using the elastic capacitary potential of the individual inclusions, we represent the tensor $V(\mathbf{x},\mathbf{y})$ as

$$V(\mathbf{x},\mathbf{y}) = -\sum_{\substack{k\neq j\\ 1\leq k\leq N}} P_\varepsilon^{(k)}(\mathbf{x})H(\mathbf{O}^{(k)},\mathbf{y}) + \mathfrak{H}_\varepsilon(\mathbf{x},\mathbf{y})\ . \tag{160}$$

Substituting (160) into (153) we have

$$H_\varepsilon(\mathbf{x},\mathbf{y}) = -\sum_{j=1}^{N} P_\varepsilon^{(j)}(\mathbf{x})H(\mathbf{O}^{(j)},\mathbf{y}) + H(\mathbf{x},\mathbf{y}) + \mathfrak{H}_\varepsilon(\mathbf{x},\mathbf{y})\ , \tag{161}$$

where $\mathfrak{H}_\varepsilon(\mathbf{x},\mathbf{y})$ is the remainder term satisfying

$$L(\partial_{\mathbf{x}})\mathfrak{H}_\varepsilon(\mathbf{x},\mathbf{y}) = 0I_2\ , \quad \mathbf{x},\mathbf{y}\in\Omega_\varepsilon\ , \tag{162}$$

$$\mathfrak{H}_\varepsilon(\mathbf{x},\mathbf{y}) = 0I_2\ , \quad \mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega_\varepsilon\ , \tag{163}$$

$$\begin{aligned}\mathfrak{H}_\varepsilon(\mathbf{x},\mathbf{y}) &= H(\mathbf{O}^{(j)},\mathbf{y}) - H(\mathbf{x},\mathbf{y})\\ &= O(\varepsilon)\ , \quad \mathbf{x}\in\partial\omega_\varepsilon^{(j)}, \mathbf{y}\in\Omega_\varepsilon, 1\leq j\leq N\ .\end{aligned} \tag{164}$$

Therefore, by Lemma 5, we have $\mathfrak{H}_\varepsilon(\mathbf{x},\mathbf{y}) = O(\varepsilon)$ uniformly in Ω_ε.

The approximation of $h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})$. We begin by writing the boundary condition (151) on $\partial\omega_\varepsilon^{(j)}$ as

$$h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = -K_2\log\varepsilon I_2 + \gamma(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)\ , \ \ \mathbf{x}\in\partial\omega_\varepsilon^{(j)}\ , \mathbf{y}\in\Omega_\varepsilon\ . \tag{165}$$

Thus we seek $h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})$ in the form

$$h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = -K_2\log\varepsilon I_2 + h^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) + \chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})\ , \tag{166}$$

for $\mathbf{x},\mathbf{y}\in\Omega_\varepsilon$, where the remainder $\chi_\varepsilon^{(j)}$ satisfies

$$L(\partial_{\mathbf{x}})\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = 0I_2\ , \quad \mathbf{x},\mathbf{y}\in\Omega_\varepsilon\ , \tag{167}$$

$$\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = K_2\log\varepsilon I_2 - h^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)\ , \quad \mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega_\varepsilon\ , \tag{168}$$

$$\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = 0I_2\ , \quad \mathbf{x}\in\partial\omega_\varepsilon^{(j)}, \mathbf{y}\in\Omega_\varepsilon\ , \tag{169}$$

$$\begin{aligned}\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) &= K_2\log\varepsilon I_2 - h^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)\ , \quad \mathbf{x}\in\partial\omega_\varepsilon^{(k)}, \mathbf{y}\in\Omega_\varepsilon\ ,\\ & \quad 1\leq k\leq N\ , k\neq j\ .\end{aligned} \tag{170}$$

Using Lemma 6, we rewrite boundary conditions (168) and (170) as

$$\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = -\gamma(\mathbf{x},\mathbf{O}^{(j)}) + \zeta^{(j)}(\boldsymbol{\eta}_j) + O(\varepsilon)\;, \quad \mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega_\varepsilon\;, \tag{171}$$

and

$$\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = -\gamma(\mathbf{x},\mathbf{O}^{(j)}) + \zeta^{(j)}(\boldsymbol{\eta}_j) + O(\varepsilon)\;, \tag{172}$$

for $\mathbf{x}\in\partial\omega_\varepsilon^{(k)}, \mathbf{y}\in\Omega_\varepsilon$, $1\le k\le N$, $k\neq j$. Then, using the elastic capacitary potential, $\chi_\varepsilon^{(j)}$ is sought in the form

$$\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = -H(\mathbf{x},\mathbf{O}^{(j)}) + (I_2 - P_\varepsilon^{(j)}(\mathbf{x}))\zeta^{(j)}(\boldsymbol{\eta}_j) + \mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})\;, \tag{173}$$

where the matrix $\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})$ satisfies

$$L(\partial_{\mathbf{x}})\mathfrak{h}_\varepsilon(\mathbf{x},\mathbf{y}) = 0I_2\;, \quad \mathbf{x},\mathbf{y}\in\Omega_\varepsilon\;, \tag{174}$$

$$\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = O(\varepsilon)\;, \quad \mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega_\varepsilon\;, \tag{175}$$

$$\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = H(\mathbf{x},\mathbf{O}^{(j)})\;, \quad \mathbf{x}\in\partial\omega_\varepsilon^{(j)}, \mathbf{y}\in\Omega_\varepsilon\;, \tag{176}$$

$$\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = -G(\mathbf{x},\mathbf{O}^{(j)}) + O(\varepsilon)\;, \quad \mathbf{x}\in\partial\omega_\varepsilon^{(k)}, \mathbf{y}\in\Omega_\varepsilon\;, 1\le k\le N\;, k\neq j\;. \tag{177}$$

From the fact that $G(\mathbf{x},\mathbf{O}^{(j)})$ and its regular part are smooth in Ω_ε, in the vicinity of the small inclusions we expand these matrices about the centres of these inclusions, in such a way that boundary conditions (176) and (177) become

$$\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = H(\mathbf{O}^{(j)},\mathbf{O}^{(j)}) + O(\varepsilon)\;, \quad \mathbf{x}\in\partial\omega_\varepsilon^{(j)}, \mathbf{y}\in\Omega_\varepsilon\;, \tag{178}$$

$$\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = -G(\mathbf{O}^{(k)},\mathbf{O}^{(j)}) + O(\varepsilon)\;, \quad \mathbf{x}\in\partial\omega_\varepsilon^{(k)}, \mathbf{y}\in\Omega_\varepsilon\;, 1\le k\le N\;, k\neq j\;. \tag{179}$$

Then, using the elastic capacitary potential, we represent $\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})$ as

$$\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = P_\varepsilon^{(j)}(\mathbf{x})H(\mathbf{O}^{(j)},\mathbf{O}^{(j)}) - \sum_{\substack{k\neq j\\ 1\le k\le N}} P_\varepsilon^{(k)}(\mathbf{x})G(\mathbf{O}^{(k)},\mathbf{O}^{(j)}) + O(\varepsilon)\;, \tag{180}$$

which is uniform by Lemma 5.

Placing (173) and (180) into (166), we obtain the approximation of $h_\varepsilon(\mathbf{x},\mathbf{y})$ in the form

$$\begin{aligned} h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) \;=\;& -K_2\log\varepsilon I_2 + h^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) - H(\mathbf{x},\mathbf{O}^{(j)}) \\ & + (I_2 - P_\varepsilon^{(j)}(\mathbf{x}))\zeta^{(j)}(\boldsymbol{\eta}_j) + P_\varepsilon^{(j)}(\mathbf{x})H(\mathbf{O}^{(j)},\mathbf{O}^{(j)}) \\ & - \sum_{\substack{k\neq j\\ 1\le k\le N}} P_\varepsilon^{(k)}(\mathbf{x})G(\mathbf{O}^{(k)},\mathbf{O}^{(j)}) + O(\varepsilon)\;. \end{aligned} \tag{181}$$

Combined formula. Now substituting (161), (181) into (145) we obtain

$$\begin{aligned} G_\varepsilon(\mathbf{x},\mathbf{y}) &= G(\mathbf{x},\mathbf{y}) + \sum_{j=1}^{N} g^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) - N\gamma(\mathbf{x},\mathbf{y}) \\ &\quad + \sum_{j=1}^{N} (I_2 - P_\varepsilon^{(j)}(\mathbf{x}))(H(\mathbf{O}^{(j)}, \mathbf{O}^{(j)}) - \zeta^{(j)}(\boldsymbol{\eta}_j) - H(\mathbf{O}^{(j)}, \mathbf{y})) \\ &\quad + \sum_{j=1}^{N} (H(\mathbf{x}, \mathbf{O}^{(j)}) + H(\mathbf{O}^{(j)}, \mathbf{y}) - H(\mathbf{O}^{(j)}, \mathbf{O}^{(j)})) \\ &\quad + \sum_{j=1}^{N} \sum_{\substack{k \neq j \\ 1 \le k \le N}} P_\varepsilon^{(k)}(\mathbf{x}) G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)}) + O(\varepsilon) \, . \end{aligned} \tag{182}$$

Using the following relation obtained from the approximation of $P_\varepsilon^{(j)}(\mathbf{x})$

$$\begin{aligned} &(A^{(j)})^{-1}(H(\mathbf{O}^{(j)}, \mathbf{O}^{(j)}) - \zeta^{(j)}(\boldsymbol{\eta}_j) - H(\mathbf{O}^{(j)}, \mathbf{y})) \\ = \; & I_2 - P_\varepsilon^{(j)T}(\mathbf{y}) + \sum_{\substack{k \neq j \\ 1 \le k \le N}} (A^{(j)})^{-1} G(\mathbf{O}^{(j)}, \mathbf{O}^{(k)}) P_\varepsilon^{(k)T}(\mathbf{y}) + O(\varepsilon (\log \varepsilon)^{-1}) \, , \end{aligned} \tag{183}$$

where $A^{(j)} = -(D^{(j)})^{-1}$, and substituting in (182) we have

$$\begin{aligned} G_\varepsilon(\mathbf{x},\mathbf{y}) &= G(\mathbf{x},\mathbf{y}) + \sum_{j=1}^{N} g^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) - N\gamma(\mathbf{x},\mathbf{y}) \\ &\quad + \sum_{j=1}^{N} (I_2 - P_\varepsilon^{(j)}(\mathbf{x})) A^{(j)} (I_2 - P_\varepsilon^{(j)T}(\mathbf{y})) \\ &\quad + \sum_{j=1}^{N} (H(\mathbf{x}, \mathbf{O}^{(j)}) + H(\mathbf{O}^{(j)}, \mathbf{y}) - H(\mathbf{O}^{(j)}, \mathbf{O}^{(j)})) \\ &\quad + \sum_{j=1}^{N} \sum_{\substack{k \neq j \\ 1 \le k \le N}} P_\varepsilon^{(k)}(\mathbf{x}) G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)}) \\ &\quad + \sum_{j=1}^{N} \sum_{\substack{k \neq j \\ 1 \le k \le N}} (I_2 - P_\varepsilon^{(j)}(\mathbf{x})) G(\mathbf{O}^{(j)}, \mathbf{O}^{(k)}) P_\varepsilon^{(k)T}(\mathbf{y}) \\ &\quad + O(\varepsilon) \, . \end{aligned} \tag{184}$$

Then, using the approximation of the elastic capacitary potential to simplify the second summand

$$\begin{aligned}
&\sum_{j=1}^{N}(I_2 - P_\varepsilon^{(j)}(\mathbf{x}))A^{(j)}(I_2 - P_\varepsilon^{(j)T}(\mathbf{y})) \\
= \quad & -\sum_{j=1}^{N}(H(\mathbf{x}, \mathbf{O}^{(j)}) + H(\mathbf{O}^{(j)}, \mathbf{y}) - H(\mathbf{O}^{(j)}, \mathbf{O}^{(j)})) \\
& -\sum_{j=1}^{N}(\zeta^{(j)}(\boldsymbol{\xi}_j) + \zeta^{(j)}(\boldsymbol{\eta}_j) - \zeta^{(\infty,j)}) - NK_2 \log \varepsilon I_2 \\
& -\sum_{j=1}^{N} \sum_{\substack{k\neq j \\ 1\leq k\leq N}} \left\{G(\mathbf{O}^{(j)}, \mathbf{O}^{(k)})P_\varepsilon^{(k)T}(\mathbf{y}) + P_\varepsilon^{(k)}(\mathbf{x})G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)})\right\} \\
& +\sum_{j=1}^{N} P_\varepsilon^{(j)}(\mathbf{x})A^{(j)}P_\varepsilon^{(j)T}(\mathbf{y}) + O(\varepsilon) \; . \qquad (185)
\end{aligned}$$

Substitution of (185) in (184) yields the formula (143). The proof is complete. □

7 Asymptotic formulae versus numerical solution for the system of two dimensional elasticity

We consider a numerical example which illustrates the accuracy of asymptotic formula of Green's tensor given by (143), for the Lamé operator in $\Omega_\varepsilon \subset \mathbb{R}^2$. As in Section 5, we shall concern ourselves only with the approximation of the regular part of (143).

7.1 Domain and asymptotic formula

The example configuration considered here is that of a half-plane with N circular inclusions. Let $\mathbb{R}^2_+$ be the half-plane

$$\mathbb{R}^2_+ := \{\mathbf{x} = (x_1, x_2) : x_1 > 0, x_2 \in \mathbb{R}\} \; , \qquad (186)$$

and let $\omega_\varepsilon^{(j)}$, $j = 1, \dots, N$ be disks of scaled radii a_j, where a_j does not exceed d defined by (93) in Section 5 and take $N = 5$. Let the point where the in-plane horizontal and vertical forces act be $\mathbf{y} = (250, 50)$. We also define ε as in Section 5.

The regular part $\mathcal{H}_\varepsilon$ of the tensor G_ε is a solution of the boundary value problem

$$L(\partial_{\mathbf{x}})\mathcal{H}_\varepsilon(\mathbf{x},\mathbf{y}) = 0I_2 \ , \quad \mathbf{x},\mathbf{y} \in \Omega_\varepsilon \ , \tag{187}$$

$$\mathcal{H}_\varepsilon(\mathbf{x},\mathbf{y}) = \gamma(\mathbf{x},\mathbf{y}) \ , \quad \mathbf{x} \in \partial\Omega_\varepsilon, \mathbf{y} \in \Omega_\varepsilon \ , \tag{188}$$

where $\Omega_\varepsilon = \mathbb{R}^2_+ \backslash \bigcup_j \bar{\omega}^{(j)}_\varepsilon$.

By (143), the asymptotic formula for the tensor $\mathcal{H}_\varepsilon$, is given as follows

$$\begin{aligned}\mathcal{H}_\varepsilon(\mathbf{x},\mathbf{y}) \ &= \ H(\mathbf{x},\mathbf{y}) + \sum_{j=1}^{N} h^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) \\ &\quad - \sum_{j=1}^{N} \left\{ P^{(j)}_\varepsilon(\mathbf{x}) A^{(j)} P^{(j)T}_\varepsilon(\mathbf{y}) - \zeta^{(j)}(\boldsymbol{\xi}_j) - \zeta^{(j)}(\boldsymbol{\eta}_j) + \zeta^{(\infty,j)} \right\} \\ &\quad + \sum_{j=1}^{N} \sum_{\substack{k \neq j \\ 1 \le k \le N}} P^{(j)}_\varepsilon(\mathbf{x}) G(\mathbf{O}^{(j)}, \mathbf{O}^{(k)}) P^{(k)T}_\varepsilon(\mathbf{y}) + O(\varepsilon) \ .\end{aligned} \tag{189}$$

The tensor $H(\mathbf{x},\mathbf{y})$. Here $H(\mathbf{x},\mathbf{y}) = [H_{ij}(\mathbf{x},\mathbf{y})]^2_{i,j=1}$, is the regular part of Green's tensor in $\mathbb{R}^2_+$, whose components, obtained from [15], are given by

$$H_{11}(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi\mu(\varkappa+1)} \left[-\varkappa \log r + \frac{(x_1+y_1)^2}{r^2} - \frac{2x_1y_1((x_2-y_2)^2 - (x_1+y_1)^2)}{\varkappa r^4} \right] \ , \tag{190}$$

$$H_{21}(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi\mu(\varkappa+1)} \left[\frac{(x_1-y_1)(x_2-y_2)}{r^2} + \frac{4x_1y_1(x_1+y_1)(x_2-y_2)}{\varkappa r^4} \right] \ , \tag{191}$$

$$H_{12}(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi\mu(\varkappa+1)} \left[\frac{(x_1-y_1)(x_2-y_2)}{r^2} - \frac{4x_1y_1(x_1+y_1)(x_2-y_2)}{\varkappa r^4} \right] \ , \tag{192}$$

$$H_{22}(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi\mu(\varkappa+1)} \left[-\varkappa \log r + \frac{(x_2-y_2)^2}{r^2} + \frac{2x_1y_1((x_1+y_1)^2 - (x_2-y_2)^2)}{\varkappa r^4} \right] \ , \tag{193}$$

where

$$r \ = \ ((x_1+y_1)^2 + (x_2-y_2)^2)^{1/2} \ , \tag{194}$$

$$\varkappa \ = \ (\lambda + 3\mu)(\lambda+\mu)^{-1} \ . \tag{195}$$

The tensor $h^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j)$. To obtain the regular part $h^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j)$ of Green's function for the exterior of the circular inclusion, we take the solution $h^{(j,D)} =$

$[h^{(j,D)}_{ik}]^2_{i,k=1}$ of the homogeneous Lamé system, which corresponds to the regular part of the displacements produced by a point force applied on the horizontal axis in an infinite plane with a circular inclusion centred at the origin (see [1], [2]). The relationship between $h^{(j)}$ and $h^{(j,D)}$ is then given by

$$\begin{aligned} h^{(j,1)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) &= \{h^{(j,D)}_{11}(\boldsymbol{\nu}_j,(b_j,0))\cos^2\theta_j - [h^{(j,D)}_{21}(\boldsymbol{\nu}_j,(b_j,0)) \\ &\quad + h^{(j,D)}_{12}(\boldsymbol{\nu}_j,(b_j,0))]\cos\theta_j\sin\theta_j + h^{(j,D)}_{22}(\boldsymbol{\nu}_j,(b_j,0))\sin^2\theta_j\}\mathbf{e}^{(1)} \\ &\quad + \{h^{(j,D)}_{21}(\boldsymbol{\nu}_j,(b_j,0))\cos^2\theta_j + [h^{(j,D)}_{11}(\boldsymbol{\nu}_j,(b_j,0)) \\ &\quad - h^{(j,D)}_{22}(\boldsymbol{\nu}_j,(b_j,0))]\cos\theta_j\sin\theta_j - h^{(j,D)}_{12}(\boldsymbol{\nu}_j,(b_j,0))\sin^2\theta_j\}\mathbf{e}^{(2)} , \end{aligned} \tag{196}$$

$$\begin{aligned} h^{(j,2)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) &= \{h^{(j,D)}_{12}(\boldsymbol{\nu}_j,(b_j,0))\cos^2\theta_j + [h^{(j,D)}_{11}(\boldsymbol{\nu}_j,(b_j,0)) \\ &\quad - h^{(j,D)}_{22}(\boldsymbol{\nu}_j,(b_j,0))]\cos\theta_j\sin\theta_j - h^{(j,D)}_{21}(\boldsymbol{\nu}_j,(b_j,0))\sin^2\theta_j\}\mathbf{e}^{(1)} \\ &\quad + \{h^{(j,D)}_{22}(\boldsymbol{\nu}_j,(b_j,0))\cos^2\theta_j + [h^{(j,D)}_{12}(\boldsymbol{\nu}_j,(b_j,0)) \\ &\quad + h^{(j,D)}_{21}(\boldsymbol{\nu}_j,(b_j,0))]\cos\theta_j\sin\theta_j + h^{(j,D)}_{11}(\boldsymbol{\nu}_j,(b_j,0))\sin^2\theta_j\}\mathbf{e}^{(2)} , \end{aligned} \tag{197}$$

where θ_j is the angle between the line that passes through the origin and the point $\boldsymbol{\eta}_j = (\eta_{j1},\eta_{j2})$ where the force is applied, and the ξ_{j1}-axis in scaled coordinates, $b_j = (\eta^2_{j1}+\eta^2_{j2})^{1/2}$, and the components of the vector $\boldsymbol{\nu}_j = (\nu_{j1},\nu_{j2})$ are represented as follows

$$\nu_{j1} = \xi_{j1}\cos\theta_j + \xi_{j2}\sin\theta_j , \tag{198}$$
$$\nu_{j2} = -\xi_{j1}\sin\theta_j + \xi_{j2}\cos\theta_j . \tag{199}$$

The components of $h^{(j,D)}(\boldsymbol{\nu}_j,(b_j,0))$ are given as follows

$$\begin{aligned} & h^{(j,D)}_{11}(\boldsymbol{\nu}_j,(b_j,0)) \\ = & -\frac{1}{4\pi\mu(\varkappa+1)}\Big\{2\varkappa\log r_1 + \frac{(\varkappa-1)(a_j^2-b_j^2)\nu_{j1}}{\varkappa b_j r_2^2} + \frac{a_j^2-2b_j^2-2\varkappa b_j^2\log(a_j b_j^{-1})}{b_j^2} \\ & + \frac{(b_j^2(\varkappa-1)(b_j^2-a_j^2)+\varkappa a_j^4)(\nu_{j1}-\frac{a_j^2}{b_j}) - \varkappa b_j^3(\nu_{j1}(\nu_{j1}-\frac{a_j^2}{b_j})-\nu_{j2}^2)}{\varkappa b_j^3 r_1^2} \\ & + \frac{a_j^2(b_j^2-a_j^2)(-(\nu_{j1}-b_j)((\nu_{j1}-\frac{a_j^2}{b_j})^2-\nu_{j2}^2)+2\nu_{j2}^2(\nu_{j1}-\frac{a_j^2}{b_j}))}{\varkappa b_j^3 r_1^4}\Big\} , \end{aligned} \tag{200}$$

$$
\begin{aligned}
& h_{21}^{(j,D)}(\boldsymbol{\nu}_j,(b_j,0)) \\
= & -\frac{1}{4\pi\mu(\varkappa+1)}\Big\{-\frac{a_j^2(b_j^2-a_j^2)\nu_{j2}((\nu_{j1}-\frac{a_j^2}{b_j})^2-\nu_{j2}^2+2(\nu_{j1}-b)(\nu_{j1}-\frac{a_j^2}{b_j}))}{\varkappa b_j^3 r_1^4} \\
& +\frac{\nu_{j2}(b_j^2a_j^2(\varkappa+1)+b_j^4(\varkappa-1)-\varkappa a_j^4-\varkappa b_j^3(2\nu_{j1}-\frac{a_j^2}{b_j}))}{\varkappa b_j^3 r_1^2} \\
& +\frac{(\varkappa-1)(a_j^2-b_j^2)\nu_{j2}}{\varkappa b_j r_2^2}\Big\}\,, \qquad (201)
\end{aligned}
$$

$$
\begin{aligned}
& h_{12}^{(j,D)}(\boldsymbol{\nu}_j,(b_j,0)) \\
= & -\frac{1}{4\pi\mu(\varkappa+1)}\Big\{\frac{a_j^2(b_j^2-a_j^2)\nu_{j2}((\nu_{j1}-\frac{a_j^2}{b_j})^2-\nu_{j2}^2+2(\nu_{j1}-b)(\nu_{j1}-\frac{a_j^2}{b_j}))}{\varkappa b_j^3 r_1^4} \\
& +\frac{\nu_{j2}(b_j^2a_j^2(\varkappa-1)+b_j^4(\varkappa+1)-\varkappa a_j^4-\varkappa b_j^3(2\nu_{j1}-\frac{a_j^2}{b_j}))}{\varkappa b_j^3 r_1^2} \\
& -\frac{(\varkappa+1)(b_j^2-a_j^2)\nu_{j2}}{\varkappa b_j r_2^2}\Big\}\,, \qquad (202)
\end{aligned}
$$

$$
\begin{aligned}
& h_{22}^{(j,D)}(\boldsymbol{\nu}_j,(b_j,0)) \\
= & -\frac{1}{4\pi\mu(\varkappa+1)}\Big\{2\varkappa\log r_1+\frac{(\varkappa+1)(b_j^2-a_j^2)\nu_{j1}}{\varkappa b_j r_2^2}-\frac{a_j^2+2\varkappa b_j^2\log(a_jb_j^{-1})}{b_j^2} \\
& +\frac{(b_j^2(\varkappa+1)(a_j^2-b_j^2)-\varkappa a_j^4)(\nu_{j1}-\frac{a_j^2}{b_j})+\varkappa b_j^3(\nu_{j1}(\nu_{j1}-\frac{a_j^2}{b_j})-\nu_{j2}^2)}{\varkappa b_j^3 r_1^2} \\
& +\frac{a_j^2(b_j^2-a_j^2)(-(\nu_{j1}-b_j)((\nu_{j1}-\frac{a_j^2}{b_j})^2-\nu_{j2}^2)+2\nu_{j2}^2(\nu_{j1}-\frac{a_j^2}{b_j}))}{\varkappa b_j^3 r_1^4}\Big\}\,, \quad (203)
\end{aligned}
$$

where a_j is the radius of the j^{th} circular inclusion in scaled coordinates,

$$
r_1=((\nu_{j1}-\frac{a_j^2}{b_j})^2+\nu_{j2}^2)^{1/2}\,, \quad r_2=(\nu_{j1}^2+\nu_{j2}^2)^{1/2}\,. \qquad (204)
$$

The tensor $\zeta^{(j)}(\boldsymbol{\xi}_j)$. The tensor $\zeta^{(j)}(\boldsymbol{\xi}_j)=[\zeta_{ik}^{(j)}(\boldsymbol{\xi}_j)]_{i,k=1}^2$, taken from [12], has components given by

$$
\zeta_{11}^{(j)}(\boldsymbol{\xi}_j)=\frac{1}{4\pi\mu(\varkappa+1)}\Big\{2\varkappa\log\left(\frac{|\boldsymbol{\xi}_j|}{a_j}\right)-\frac{\xi_{j1}^2-\xi_{j2}^2}{|\boldsymbol{\xi}_j|^2}+\frac{a_j^2(\xi_{j1}^2-\xi_{j2}^2)}{|\boldsymbol{\xi}_j|^4}\Big\}\,, \quad (205)
$$

$$\zeta_{12}^{(j)}(\boldsymbol{\xi}_j) = \zeta_{21}^{(j)}(\boldsymbol{\xi}_j) = \frac{1}{2\pi\mu(\varkappa+1)}\left\{\frac{a_j^2\xi_{j1}\xi_{j2}}{|\boldsymbol{\xi}_j|^4} - \frac{\xi_{j1}\xi_{j2}}{|\boldsymbol{\xi}_j|^2}\right\}, \quad (206)$$

$$\zeta_{22}^{(j)}(\boldsymbol{\xi}_j) = \frac{1}{4\pi\mu(\varkappa+1)}\left\{2\varkappa\log\left(\frac{|\boldsymbol{\xi}_j|}{a_j}\right) + \frac{\xi_{j1}^2 - \xi_{j2}^2}{|\boldsymbol{\xi}_j|^2} - \frac{a_j^2(\xi_{j1}^2 - \xi_{j2}^2)}{|\boldsymbol{\xi}_j|^4}\right\}. \quad (207)$$

The constant matrix $\zeta^{(\infty,j)}$. The constant matrix $\zeta^{(\infty,j)}$ present in the asymptotics of the matrix function $\zeta^{(j)}$ at infinity (see (119)). For the case of a circular insert in an infinite plane it has the form

$$\zeta^{(\infty,j)} = \frac{1}{4\pi\mu(\varkappa+1)}\begin{pmatrix} -2\varkappa\log a_j + 1 & 0 \\ 0 & -2\varkappa\log a_j + 1 \end{pmatrix}. \quad (208)$$

The elastic capacitary potential. For the elastic capacitary potential $P_\varepsilon^{(j)}$ we shall use the solution of the system obtained from (126).

7.2 Numerical simulation

Now we discuss the comparison of the computations based on the asymptotic formula for the regular part (189) against those given in FEMLAB. This is carried out as follows: we compute the total displacements for the first and second columns of the regular part using the FEMLAB kernel, then we perform the same calculations using the leading order part of our approximation (189). The comparison is then made by taking the absolute error between the computations produced in FEMLAB and those produced by our approximation. Since it is not possible to program unbounded domains in FEMLAB, we replace the half-plane of our example by a sufficiently large semidisk of radius 5000 throughout the numerical computations.

For these experiments, we take Young's modulus to be 1.4×10^{11} Nm^{-2} and Poisson's ratio to be 0.25, which corresponds to the case of Cast Iron. In this case the elastic moduli are $\lambda = \mu = 5.6 \times 10^{10}$ Nm^{-2}.

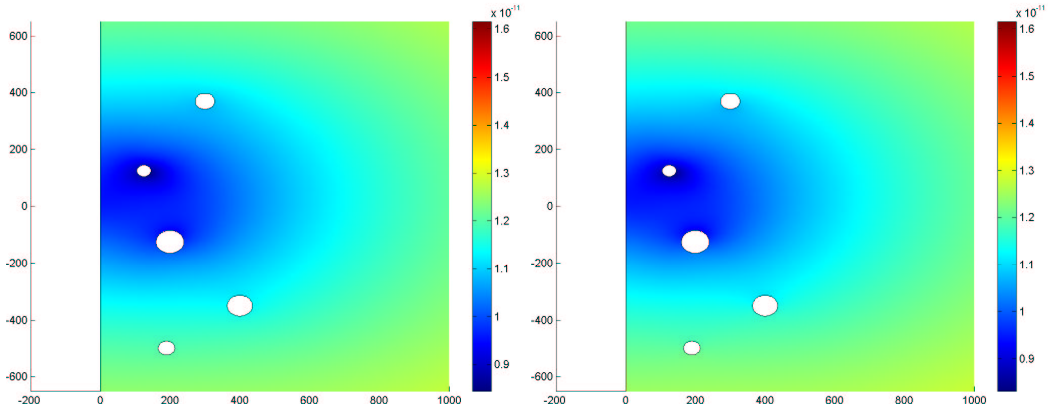
7.2.1 The case of five circular inclusions

Let $N = 5$ and $\omega_\varepsilon^{(j)}$, $j = 1,2,3,4,5$ be circular inclusions contained in the domain Ω_ε. The inclusions $\omega_\varepsilon^{(j)}$, $j = 1,2,3,4,5$ have centres $\mathbf{O}^{(1)} = (125,125)$, $\mathbf{O}^{(2)} = (200,-125)$, $\mathbf{O}^{(3)} = (300,370)$, $\mathbf{O}^{(4)} = (190,-500)$, $\mathbf{O}^{(5)} = (400,-350)$ and scaled radii 62.5, 125, 87.5, 75, 112.5, respectively. We consider the situation when $\varepsilon = 0.32$.

Fig 7 a), b) shows the 2D plot of the numerical solution given in FEMLAB and that produced according to the asymptotic formula for the first column of $\mathcal{H}_\varepsilon$, when Ω_ε contains 5 circular inclusions. Fig 8 a), b) shows the same

2D plots, which are done for the second column of $\mathcal{H}_\varepsilon$. Then we compare the computations given by the asymptotic formula and the method of finite elements for the first and second column of $\mathcal{H}_\varepsilon$ by computing the absolute error between the data, the results are shown in Fig 9 a), b).

The error stays within the range predicted by the asymptotic theory.



a) b)

Figure 7: a) Numerical solution produced in FEMLAB on a mesh containing 66480 elements, b) Computations based on the asymptotic formula for the first column of $\mathcal{H}_\varepsilon$, when $\varepsilon = 0.32$.

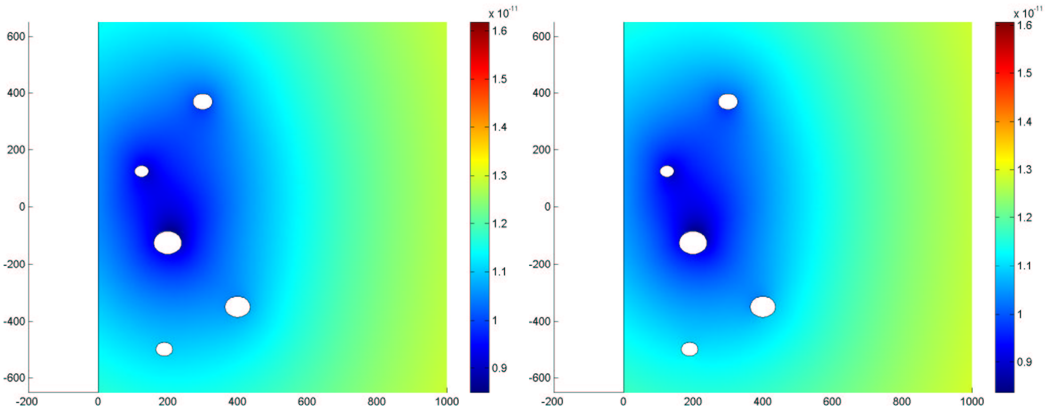
8 Green's Matrix for a 3-dimensional domain with several small inclusions

Now that the study of the approximation of Green's kernel for the situations of anti-plane shear and plane strain of elasticity have been considered, we now formulate and produce an approximation of Green's matrix for the system of elasticity in a 3-dimensional domain with multiple inclusions.

8.1 Green's tensors for model domains in three dimensions

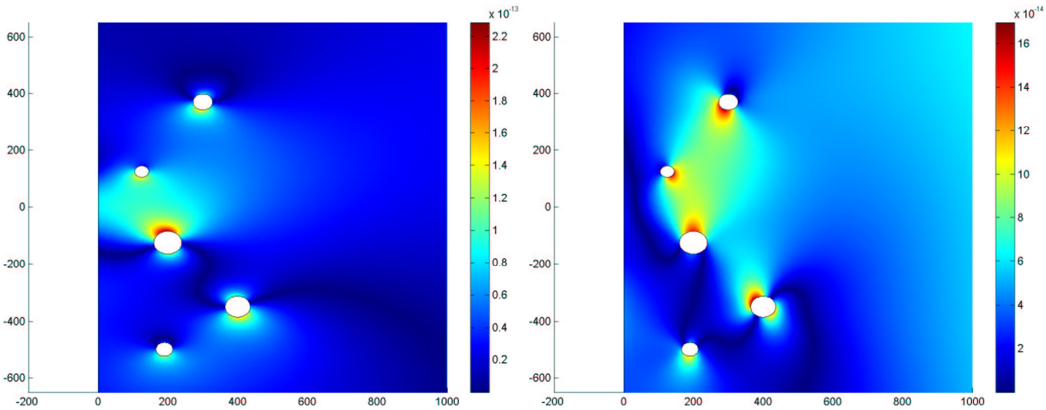
Let $G(\mathbf{x}, \mathbf{y}) = [G^{(1)}(\mathbf{x}, \mathbf{y}), G^{(2)}(\mathbf{x}, \mathbf{y}), G^{(3)}(\mathbf{x}, \mathbf{y})]$ and $g^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) = [g^{(j,1)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j)$, $g^{(j,2)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j)$, $g^{(j,3)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j)]$ denote Green's tensors for the Lamé operator

$$L := \mu\Delta + (\lambda + \mu)\nabla(\nabla\cdot\ )\ , \tag{209}$$



a) b)

Figure 8: a) Numerical solution produced in FEMLAB on a mesh containing 66480 elements, b) Computations based on the asymptotic formula for the second column of $\mathcal{H}_\varepsilon$, when $\varepsilon = 0.32$.



a) b)

Figure 9: a) Absolute error between computations given in FEMLAB and those by the asymptotic formula for a) the first column and b) the second column of $\mathcal{H}_\varepsilon$ when $\varepsilon = 0.32$ in the vicinity of the inclusions. In a) the maximum absolute error is 2.285×10^{-13}, which occurs on boundary of the inclusion with centre (200, -125), and in b) the maximum absolute error is 1.697×10^{-13}, which occurs on boundary of the inclusion with centre (400, -350).

in the sets Ω and $C\bar{\omega}^{(j)} = \mathbb{R}^3 \backslash \bar{\omega}^{(j)}$, $j = 1, \dots, N$, respectively. In the present section, the tensor G solves the following problem

$$L(\partial_{\mathbf{x}})G(\mathbf{x},\mathbf{y}) + \delta(\mathbf{x}-\mathbf{y})I_3 = 0I_3 \ , \quad \mathbf{x},\mathbf{y} \in \Omega \ , \tag{210}$$

$$G(\mathbf{x},\mathbf{y}) = 0I_3 \ , \quad \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega \ , \tag{211}$$

and the tensors $g^{(j)}$ are solutions of

$$L(\partial_{\boldsymbol{\xi}_j})g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) + \delta(\boldsymbol{\xi}_j-\boldsymbol{\eta}_j)I_3 = 0I_3 \ , \quad \boldsymbol{\xi}_j,\boldsymbol{\eta}_j \in C\bar{\omega}^{(j)} \ , \tag{212}$$

$$g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) = 0I_3 \ , \quad \boldsymbol{\xi}_j \in \partial C\bar{\omega}^{(j)}, \boldsymbol{\eta}_j \in C\bar{\omega}^{(j)} \ , \tag{213}$$

$$g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) \to 0I_3 \quad \text{as} \quad |\boldsymbol{\xi}_j| \to \infty, \boldsymbol{\eta}_j \in C\bar{\omega}^{(j)} \ , \tag{214}$$

for $j = 1, \dots, N$.

We represent $G(\mathbf{x},\mathbf{y})$ and $g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)$ as

$$G(\mathbf{x},\mathbf{y}) = \Gamma(\mathbf{x},\mathbf{y}) - H(\mathbf{x},\mathbf{y}) \ , \tag{215}$$

and

$$g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) = \Gamma(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) - h^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) \ , \tag{216}$$

where $\Gamma(\mathbf{x},\mathbf{y}) = [\Gamma_{mn}(\mathbf{x},\mathbf{y})]_{m,n=1}^3$, is the fundamental solution of the Lamé operator in three dimensions, whose entries are given by

$$\Gamma_{mn}(\mathbf{x},\mathbf{y}) = (8\pi\mu(\lambda+2\mu)|\mathbf{x}-\mathbf{y}|)^{-1}((\lambda+\mu)(x_m-y_m)(x_n-y_n)|\mathbf{x}-\mathbf{y}|^{-2}+(\lambda+3\mu)\delta_{mn}) \ , \tag{217}$$

and H, $h^{(j)}$ are the regular parts of G, $g^{(j)}$, $j = 1, \dots, N$, respectively.

8.2 Auxiliary matrix functions in three dimensions

8.2.1 The elastic capacitary matrix

We denote by $P^{(j)}(\boldsymbol{\xi}_j) = [P^{(j,1)}(\boldsymbol{\xi}_j), P^{(j,2)}(\boldsymbol{\xi}_j), P^{(j,3)}(\boldsymbol{\xi}_j)]$ the elastic capacitary potential matrix of the set $\omega^{(j)}$, which is defined as a solution of

$$L(\partial_{\boldsymbol{\xi}_j})P^{(j)}(\boldsymbol{\xi}_j) = 0I_3 \ , \quad \boldsymbol{\xi}_j \in C\bar{\omega}^{(j)} \ , \tag{218}$$

$$P^{(j)}(\boldsymbol{\xi}_j) = I_3 \ , \quad \boldsymbol{\xi}_j \in \partial\omega^{(j)} \ , \tag{219}$$

$$P^{(j)}(\boldsymbol{\xi}_j) \to 0I_3 \quad \text{as} \quad |\boldsymbol{\xi}_j| \to \infty \ , \tag{220}$$

for $j = 1, \dots, N$.

Let $B^{(j)} = [B^{(j,i)}]_{i=1}^3$ be the elastic capacity matrix for the set $\omega^{(j)}$, for $j = 1, \dots, N$. This matrix was introduced and its properties where studied in [10]. In particular, it was shown that this matrix is symmetric.

For the proof of the following Lemma, we refer to [10] subsection 4.2, Lemmas 4 and 6.

Lemma 9 *i) If* $|\boldsymbol{\xi}_j| \geq 2$*, then for* $P^{(j,i)}$*,* $i = 1, 2, 3$*, the following estimate holds*

$$|P^{(j,i)}(\boldsymbol{\xi}_j) - \Gamma(\boldsymbol{\xi}_j, \mathbf{O})B^{(j,i)}| \leq \text{const}\, |\boldsymbol{\xi}_j|^{-2} \,, \tag{221}$$

where $B^{(j,i)}$ *are the columns of the symmetric elastic capacity matrix* $B^{(j)}$ *of the set* $\omega^{(j)}$*.*

ii) The columns $P^{(j,i)}$*,* $i = 1, 2$ *or* 3*, of the elastic capacitary potential of the set* $\omega^{(j)}$*,* $j = 1, \dots, N$*, satisfy the inequality*

$$\sup_{\boldsymbol{\xi}_j \in C\bar{\omega}^{(j)}} \{|\boldsymbol{\xi}_j||P^{(j,i)}(\boldsymbol{\xi}_j)|\} \leq \text{const} \,, \quad j = 1, \dots, N \,. \tag{222}$$

8.2.2 An estimate for the regular part $h^{(j)}$ of Green's tensor in the unbounded domain

Now we present an asymptotic expansion for the regular part $h^{(j)}$ of Green's tensor $g^{(j)}$, whose proof is found in [10], Lemma 11.

Lemma 10 *For all* $\boldsymbol{\eta}_j \in C\bar{\omega}^{(j)}$ *and* $\boldsymbol{\xi}_j$ *with* $|\boldsymbol{\xi}_j| > 2$*, the following estimate for the columns* $h^{(j,i)}$ *,* $i = 1, 2,$ *or* 3*, of the regular part of* $g^{(j,i)}$ *holds*

$$|h^{(j,i)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) - \Gamma(\boldsymbol{\xi}_j, \mathbf{O})P^{(j,i)T}(\boldsymbol{\eta}_j)| \leq \text{const}\, |\boldsymbol{\xi}_j|^{-2}|\boldsymbol{\eta}_j|^{-1} \,, \tag{223}$$

where $j = 1, \dots, N$*.*

8.3 A uniform asymptotic formula for Green's tensor in a 3-dimensional domain with several inclusions

Now we present the main result concerning the approximation of the matrix G_ε, for a 3-dimensional domain with multiple inclusions.

Theorem 4 *Green's tensor* G_ε *for the Lamé operator in the domain* $\Omega_\varepsilon \subset \mathbb{R}^3$

admits the representation

$$\begin{aligned}
G_\varepsilon(\mathbf{x},\mathbf{y}) &= G(\mathbf{x},\mathbf{y})+\varepsilon^{-1}\sum_{j=1}^{N} g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)-N\Gamma(\mathbf{x},\mathbf{y})\\
&+\sum_{j=1}^{N}\big\{P^{(j)}(\boldsymbol{\xi}_j)H(\mathbf{O}^{(j)},\mathbf{y})+H(\mathbf{x},\mathbf{O}^{(j)})P^{(j)T}(\boldsymbol{\eta}_j)\\
&-P^{(j)}(\boldsymbol{\xi}_j)H(\mathbf{O}^{(j)},\mathbf{O}^{(j)})P^{(j)T}(\boldsymbol{\eta}_j)-\varepsilon H(\mathbf{x},\mathbf{O}^{(j)})B^{(j)}H(\mathbf{O}^{(j)},\mathbf{y})\big\}\\
&+\sum_{j=1}^{N}\sum_{\substack{k\neq j\\ 1\le k\le N}} P^{(k)}(\boldsymbol{\xi}_k)G(\mathbf{O}^{(k)},\mathbf{O}^{(j)})P^{(j)T}(\boldsymbol{\eta}_j)\\
&+O\left(\sum_{j=1}^{N}\varepsilon^2(\min\{|\mathbf{x}-\mathbf{O}^{(j)}|\,,|\mathbf{y}-\mathbf{O}^{(j)}|\})^{-1}\right)\,, \qquad (224)
\end{aligned}$$

uniformly with respect to $(\mathbf{x},\mathbf{y})\in\Omega_\varepsilon\times\Omega_\varepsilon$.

As in the preceding sections, we seek G_ε in the form

$$G_\varepsilon(\mathbf{x},\mathbf{y})=\Gamma(\mathbf{x},\mathbf{y})-H_\varepsilon(\mathbf{x},\mathbf{y})-\sum_{j=1}^{N}h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})\,, \qquad (225)$$

where the tensors $H_\varepsilon(\mathbf{x},\mathbf{y})$ and $h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})$ are solutions of the problems

$$L(\partial_{\mathbf{x}})H_\varepsilon(\mathbf{x},\mathbf{y})=0I_3\,,\quad \mathbf{x},\mathbf{y}\in\Omega_\varepsilon\,, \qquad (226)$$

$$H_\varepsilon(\mathbf{x},\mathbf{y})=\Gamma(\mathbf{x},\mathbf{y})\,,\quad \mathbf{x}\in\partial\Omega,\mathbf{y}\in\Omega_\varepsilon\,, \qquad (227)$$

$$H_\varepsilon(\mathbf{x},\mathbf{y})=0I_3\,,\quad \mathbf{x}\in\partial\omega_\varepsilon^{(j)},\mathbf{y}\in\Omega_\varepsilon,1\le j\le N\,, \qquad (228)$$

and

$$L(\partial_{\mathbf{x}})h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})=0I_3\,,\quad \mathbf{x},\mathbf{y}\in\Omega_\varepsilon\,, \qquad (229)$$

$$h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})=0I_3\,,\quad \mathbf{x}\in\partial\Omega,\mathbf{y}\in\Omega_\varepsilon\,, \qquad (230)$$

$$h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})=\Gamma(\mathbf{x},\mathbf{y})\,,\quad \mathbf{x}\in\partial\omega_\varepsilon^{(j)},\mathbf{y}\in\Omega_\varepsilon\,, \qquad (231)$$

$$h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})=0I_3\,,\quad \mathbf{x}\in\partial\omega_\varepsilon^{(k)},\mathbf{y}\in\Omega_\varepsilon\,,1\le k\le N\,,k\neq j\,. \qquad (232)$$

The approximation of $H_\varepsilon(\mathbf{x},\mathbf{y})$. Consider the tensor $H_\varepsilon(\mathbf{x},\mathbf{y})-H(\mathbf{x},\mathbf{y})$. This satisfies the homogeneous Lamé equation and has zero boundary data for $\mathbf{x}\in\partial\Omega,\mathbf{y}\in\Omega_\varepsilon$. For $\mathbf{x}\in\partial\omega_\varepsilon^{(j)},\mathbf{y}\in\Omega_\varepsilon$, $1\le j\le N$, this matrix is equal to $-H(\mathbf{x},\mathbf{y})$, whose leading order part is $-H(\mathbf{O}^{(j)},\mathbf{y})$. Then we may approximate H_ε, using the elastic capacitary potential, by

$$H_\varepsilon(\mathbf{x},\mathbf{y}) - H(\mathbf{x},\mathbf{y}) = -\sum_{j=1}^{N} P^{(j)}(\boldsymbol{\xi}_j)H(\mathbf{O}^{(j)},\mathbf{y}) + \mathfrak{S}_\varepsilon(\mathbf{x},\mathbf{y})\ , \tag{233}$$

where the remainder term $\mathfrak{S}_\varepsilon$ on the right is a solution of the homogeneous Lamé equation, is $O(\varepsilon)$ for $\mathbf{x} \in \partial\omega_\varepsilon^{(j)}, \mathbf{y} \in \Omega_\varepsilon$, $1 \le j \le N$ and by Lemma 9 *i)* the leading order part of $\mathfrak{S}_\varepsilon$ is

$$\sum_{j=1}^{N} \varepsilon\Gamma(\mathbf{x},\mathbf{O}^{(j)})B^{(j)}H(\mathbf{O}^{(j)},\mathbf{y}) \text{ for } \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega_\varepsilon\ . \tag{234}$$

Then the approximation of $\mathfrak{S}_\varepsilon(\mathbf{x},\mathbf{y})$ may be given by

$$\mathfrak{S}_\varepsilon(\mathbf{x},\mathbf{y}) = \sum_{j=1}^{N} \varepsilon H(\mathbf{x},\mathbf{O}^{(j)})B^{(j)}H(\mathbf{O}^{(j)},\mathbf{y}) + \mathfrak{H}_\varepsilon(\mathbf{x},\mathbf{y})\ , \tag{235}$$

then upon substitution of this into (233) we obtain the following approximation for H_ε

$$\begin{aligned} H_\varepsilon(\mathbf{x},\mathbf{y}) \;\; &= \;\; H(\mathbf{x},\mathbf{y}) - \sum_{j=1}^{N} \{P^{(j)}(\boldsymbol{\xi}_j)H(\mathbf{O}^{(j)},\mathbf{y}) \\ &\qquad - \varepsilon H(\mathbf{x},\mathbf{O}^{(j)})B^{(j)}H(\mathbf{O}^{(j)},\mathbf{y})\} + \mathfrak{H}_\varepsilon(\mathbf{x},\mathbf{y})\ , \end{aligned} \tag{236}$$

where $\mathfrak{H}_\varepsilon(\mathbf{x},\mathbf{y})$ represents the remainder given by this approximation.

The approximation of $h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})$*.* The matrix

$$W^{(j)}(\mathbf{x},\mathbf{y}) = h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) - \varepsilon^{-1}h^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)\ , \tag{237}$$

satisfies the homogeneous Lamé equation, is equal to $0I_3$ on the boundary of the inclusion $\partial\omega_\varepsilon^{(j)}$ and

$$W^{(j)}(\mathbf{x},\mathbf{y}) = -\varepsilon^{-1}h^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)\ , \quad \mathbf{x} \in \partial\Omega\ , \mathbf{y} \in \Omega_\varepsilon\ , \tag{238}$$

$$W^{(j)}(\mathbf{x},\mathbf{y}) = -\varepsilon^{-1}h^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)\ , \quad \mathbf{x} \in \partial\omega_\varepsilon^{(k)}\ , \mathbf{y} \in \Omega_\varepsilon\ , k \ne j\ , 1 \le k \le N\ . \tag{239}$$

By Lemma 10, the boundary conditions (238), (239) are equivalent to

$$W^{(j)}(\mathbf{x},\mathbf{y}) = -\Gamma(\mathbf{x},\mathbf{O}^{(j)})P^{(j)T}(\boldsymbol{\eta}_j) + O(\varepsilon^2|\mathbf{y}-\mathbf{O}^{(j)}|^{-1})\ , \quad \mathbf{x} \in \partial\Omega\ , \mathbf{y} \in \Omega_\varepsilon\ , \tag{240}$$

$$W^{(j)}(\mathbf{x},\mathbf{y}) = -\Gamma(\mathbf{x},\mathbf{O}^{(j)})P^{(j)T}(\boldsymbol{\eta}_j) + O(\varepsilon^2|\mathbf{y}-\mathbf{O}^{(j)}|^{-1}) \, , \tag{241}$$

for $\mathbf{x} \in \partial\omega_\varepsilon^{(k)}$, $\mathbf{y} \in \Omega_\varepsilon$, $k \neq j$, $1 \leq k \leq N$.

Then the function $W^{(j)}(\mathbf{x},\mathbf{y})$ is sought in the form

$$W^{(j)}(\mathbf{x},\mathbf{y}) = -H(\mathbf{x},\mathbf{O}^{(j)})P^{(j)T}(\boldsymbol{\eta}_j) + \chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) \, , \tag{242}$$

where the matrix $\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})$ is a solution of the boundary value problem

$$L(\partial_{\mathbf{x}})\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = 0I_3 \, , \quad \mathbf{x},\mathbf{y} \in \Omega_\varepsilon \, , \tag{243}$$

$$\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = O(\varepsilon^2|\mathbf{y}-\mathbf{O}^{(j)}|^{-1}) \, , \quad \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega_\varepsilon \, , \tag{244}$$

$$\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = H(\mathbf{x},\mathbf{O}^{(j)})P^{(j)T}(\boldsymbol{\eta}_j) \, , \quad \mathbf{x} \in \partial\omega_\varepsilon^{(j)}, \mathbf{y} \in \Omega_\varepsilon \, , \tag{245}$$

$$\begin{aligned} \chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) \;\; &= \;\; -G(\mathbf{x},\mathbf{O}^{(j)})P^{(j)T}(\boldsymbol{\eta}_j) + O(\varepsilon^2|\mathbf{y}-\mathbf{O}^{(j)}|^{-1}) \, , \\ &\quad\;\; \mathbf{x} \in \partial\omega_\varepsilon^{(k)}, \mathbf{y} \in \Omega_\varepsilon \, , 1 \leq k \leq N \, , k \neq j \, . \end{aligned} \tag{246}$$

Since the tensor $G(\mathbf{x},\mathbf{O}^{(j)})$ and the regular part $H(\mathbf{x},\mathbf{y})$ of Green's tensor for the domain Ω, have smooth components for $\mathbf{x}$, $\mathbf{y} \in \Omega_\varepsilon$, then on $\partial\omega_\varepsilon^{(j)}$ we may expand these tensors about the centres of $\omega_\varepsilon^{(j)}$ ($1 \leq j \leq N$). Thus from (245), (246) we obtain

$$\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = H(\mathbf{O}^{(j)},\mathbf{O}^{(j)})P^{(j)T}(\boldsymbol{\eta}_j) + O(\varepsilon^2|\mathbf{y}-\mathbf{O}^{(j)}|^{-1}) \, , \tag{247}$$

for $\mathbf{x} \in \partial\omega_\varepsilon^{(j)}, \mathbf{y} \in \Omega_\varepsilon$, and

$$\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) = -G(\mathbf{O}^{(k)},\mathbf{O}^{(j)})P^{(j)T}(\boldsymbol{\eta}_j) + O(\varepsilon^2|\mathbf{y}-\mathbf{O}^{(j)}|^{-1}) \, , \tag{248}$$

for $\mathbf{x} \in \partial\omega_\varepsilon^{(k)}, \mathbf{y} \in \Omega_\varepsilon$, $1 \leq k \leq N$, $k \neq j$.

However, (247) and (248) are not small on the exterior boundary $\partial\Omega$. Therefore, using the elastic capacitary potential we represent $\chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})$ as

$$\begin{aligned} \chi_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) \;\; &= \;\; P^{(j)}(\boldsymbol{\xi}_j)H(\mathbf{O}^{(j)},\mathbf{O}^{(j)})P^{(j)T}(\boldsymbol{\eta}_j) \\ &\quad - \sum_{\substack{k\neq j \\ 1\leq k\leq N}} P^{(k)}(\boldsymbol{\xi}_k)G(\mathbf{O}^{(k)},\mathbf{O}^{(j)})P^{(j)T}(\boldsymbol{\eta}_j) \\ &\quad + \mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) \, , \end{aligned} \tag{249}$$

where the matrix $\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})$ is the remainder term.

Collecting (242) and (249) in (237), we have the following approximation for the tensor $h_\varepsilon^{(j)}$

$$\begin{aligned} h_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) &= \varepsilon^{-1}h^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) - H(\mathbf{x},\mathbf{O}^{(j)})P^{(j)T}(\boldsymbol{\eta}_j) \\ &+P^{(j)}(\boldsymbol{\xi}_j)H(\mathbf{O}^{(j)},\mathbf{O}^{(j)})P^{(j)T}(\boldsymbol{\eta}_j) \\ &- \sum_{\substack{k\neq j \\ 1\le k\le N}} P^{(k)}(\boldsymbol{\xi}_k)G(\mathbf{O}^{(k)},\mathbf{O}^{(j)})P^{(j)T}(\boldsymbol{\eta}_j) \\ &+\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) \ . \end{aligned} \tag{250}$$

Combined formula. Substituting (236) and (250) in (225) we obtain

$$\begin{aligned} G_\varepsilon(\mathbf{x},\mathbf{y}) &= G(\mathbf{x},\mathbf{y}) + \varepsilon^{-1}\sum_{j=1}^{N} g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) - N\Gamma(\mathbf{x},\mathbf{y}) \\ &+\sum_{j=1}^{N}\big\{P^{(j)}(\boldsymbol{\xi}_j)H(\mathbf{O}^{(j)},\mathbf{y}) + H(\mathbf{x},\mathbf{O}^{(j)})P^{(j)T}(\boldsymbol{\eta}_j) \\ &-P^{(j)}(\boldsymbol{\xi}_j)H(\mathbf{O}^{(j)},\mathbf{O}^{(j)})P^{(j)T}(\boldsymbol{\eta}_j) - \varepsilon H(\mathbf{x},\mathbf{O}^{(j)})B^{(j)}H(\mathbf{O}^{(j)},\mathbf{y})\big\} \\ &+\sum_{j=1}^{N}\sum_{\substack{k\neq j \\ 1\le k\le N}} P^{(k)}(\boldsymbol{\xi}_k)G(\mathbf{O}^{(k)},\mathbf{O}^{(j)})P^{(j)T}(\boldsymbol{\eta}_j) \\ &+R_\varepsilon(\mathbf{x},\mathbf{y}) \ , \end{aligned} \tag{251}$$

where the matrix R_ε represents the combination of the remainder terms $\mathfrak{H}_\varepsilon$ and $\mathfrak{h}_\varepsilon^{(j)}$, $j = 1,\dots,N$, given in the approximations (236) and (250), respectively.

We now give a rigorous proof of (224), including the remainder estimate.

8.3.1 Proof of Theorem 4

From (251), the columns $R_\varepsilon^{(k)}(\mathbf{x},\mathbf{y})$, $k = 1,2,3$ of the remainder, satisfy the boundary value problem

$$\mu\Delta_{\mathbf{x}}R_\varepsilon^{(k)}(\mathbf{x},\mathbf{y}) + (\lambda+\mu)\nabla_{\mathbf{x}}(\nabla_{\mathbf{x}}\cdot R_\varepsilon^{(k)}(\mathbf{x},\mathbf{y})) = \mathbf{O} \ , \quad \mathbf{x},\mathbf{y}\in\Omega_\varepsilon \ , \tag{252}$$

$$
\begin{aligned}
R_\varepsilon^{(k)}(\mathbf{x},\mathbf{y}) &= \varepsilon^{-1}\sum_{j=1}^{N} h^{(j,k)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) - \sum_{j=1}^{N}\{P^{(j)}(\boldsymbol{\xi}_j)H^{(k)}(\mathbf{O}^{(j)},\mathbf{y}) \\
&\quad + H(\mathbf{x},\mathbf{O}^{(j)})P^{(j,k)T}(\boldsymbol{\eta}_j) - \varepsilon H(\mathbf{x},\mathbf{O}^{(j)})B^{(j)}H^{(k)}(\mathbf{O}^{(j)},\mathbf{y}) \\
&\quad - P^{(j)}(\boldsymbol{\xi}_j)H(\mathbf{O}^{(j)},\mathbf{O}^{(j)})P^{(j,k)T}(\boldsymbol{\eta}_j)\} \\
&\quad - \sum_{j=1}^{N}\sum_{\substack{l\neq j\\ 1\leq l\leq N}} P^{(l)}(\boldsymbol{\xi}_l)G(\mathbf{O}^{(l)},\mathbf{O}^{(j)})P^{(j,k)T}(\boldsymbol{\eta}_j)\ , \\
&\quad \text{for } \mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega_\varepsilon\ , \qquad (253)
\end{aligned}
$$

$$
\begin{aligned}
R_\varepsilon^{(k)}(\mathbf{x},\mathbf{y}) &= H^{(k)}(\mathbf{x},\mathbf{y}) - H^{(k)}(\mathbf{O}^{(m)},\mathbf{y}) + \varepsilon^{-1}\sum_{\substack{j\neq m\\ 1\leq j\leq N}} h^{(j,k)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) \\
&\quad - \{H(\mathbf{x},\mathbf{O}^{(m)}) - H(\mathbf{O}^{(m)},\mathbf{O}^{(m)})\}P^{(m,k)T}(\boldsymbol{\eta}_m) \\
&\quad - \sum_{\substack{j\neq m\\ 1\leq j\leq N}}\{P^{(j)}(\boldsymbol{\xi}_j)H^{(k)}(\mathbf{O}^{(j)},\mathbf{y}) + H(\mathbf{x},\mathbf{O}^{(j)})P^{(j,k)T}(\boldsymbol{\eta}_j) \\
&\quad - P^{(j)}(\boldsymbol{\xi}_j)H(\mathbf{O}^{(j)},\mathbf{O}^{(j)})P^{(j,k)T}(\boldsymbol{\eta}_j)\} \\
&\quad + \sum_{j=1}^{N}\varepsilon H(\mathbf{x},\mathbf{O}^{(j)})B^{(j)}H^{(k)}(\mathbf{O}^{(j)},\mathbf{y}) \\
&\quad - \sum_{\substack{j\neq m\\ 1\leq j\leq N}} G(\mathbf{O}^{(m)},\mathbf{O}^{(j)})P^{(j,k)T}(\boldsymbol{\eta}_j) \\
&\quad - \sum_{j=1}^{N}\sum_{\substack{l\neq j\neq m\\ 1\leq l\leq N}} P^{(l)}(\boldsymbol{\xi}_l)G(\mathbf{O}^{(l)},\mathbf{O}^{(j)})P^{(j,k)T}(\boldsymbol{\eta}_j) \\
&\quad \text{for } \mathbf{x}\in\partial\omega_\varepsilon^{(m)}, \mathbf{y}\in\Omega_\varepsilon\ , 1\leq m\leq N. \qquad (254)
\end{aligned}
$$

The components of $H^{(k)}(\mathbf{x},\mathbf{O}^{(j)})$ and $H^{(k)}(\mathbf{O}^{(j)},\mathbf{y})$ of H are bounded in Ω and the components of $H^{(k)}(\mathbf{x},\mathbf{O}^{(j)})$ are bounded on $\partial\Omega$. They are also bounded for $\mathbf{x}\in\partial\omega_\varepsilon^{(m)}$, $\mathbf{y}\in\Omega_\varepsilon$, $1\leq m\leq N$. Therefore, the norms of the terms

$$
\sum_{\substack{j\neq m\\ 1\leq j\leq N}} \varepsilon H(\mathbf{x},\mathbf{O}^{(j)})B^{(j)}H^{(k)}(\mathbf{O}^{(k)},\mathbf{y})\ ,
$$

are bounded by const ε in (253), (254).

By Lemma 9 ii), since the entries of $P^{(j)}(\boldsymbol{\eta}_j)$ are bounded, we have

$$
|H^{(k)}(\mathbf{x},\mathbf{y}) - H^{(k)}(\mathbf{O}^{(m)},\mathbf{y}) - (H(\mathbf{x},\mathbf{O}^{(m)}) - H(\mathbf{O}^{(m)},\mathbf{O}^{(m)}))P^{(m,k)T}(\boldsymbol{\eta}_m)|
$$

$$\leq \text{const } \varepsilon \, , \quad \text{for } \mathbf{x} \in \partial\omega_\varepsilon^{(m)}, \mathbf{y} \in \Omega_\varepsilon \, , 1 \leq m \leq N \, . \tag{255}$$

Then using the estimate given in Lemma 10 for the columns of $h^{(j)}$, $j \neq m$, we have

$$\Big| \sum_{\substack{j \neq m \\ 1 \leq j \leq N}} \{\varepsilon^{-1} h^{(j,k)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) - H(\mathbf{x}, \mathbf{O}^{(j)}) P^{(j,k)T}(\boldsymbol{\eta}_j) - G(\mathbf{O}^{(m)}, \mathbf{O}^{(j)}) P^{(j,k)T}(\boldsymbol{\eta}_j)\} \Big|$$

$$\begin{aligned} &\leq \Big| \sum_{\substack{j \neq m \\ 1 \leq j \leq N}} \{G(\mathbf{x}, \mathbf{O}^{(j)}) - G(\mathbf{O}^{(m)}, \mathbf{O}^{(j)})\} P^{(j,k)T}(\boldsymbol{\eta}^{(j)}) \Big| + \text{const} \sum_{\substack{j \neq m \\ 1 \leq j \leq N}} \varepsilon^2 |\mathbf{y} - \mathbf{O}^{(j)}|^{-1} \\ &\leq \text{const} \sum_{\substack{j \neq m \\ 1 \leq j \leq N}} \varepsilon^2 |\mathbf{y} - \mathbf{O}^{(j)}|^{-1} \, . \end{aligned} \tag{256}$$

for $\mathbf{x} \in \partial\omega_\varepsilon^{(m)}$, $\mathbf{y} \in \Omega_\varepsilon$.

Finally, using the estimate for $P^{(j)}$ of Lemma 9 *i*) for $j \neq m$, and Lemma 9 *ii*), also the fact that the components of H and $G(\mathbf{O}^{(l)}, \mathbf{O}^{(j)})$, $j \neq l$ are bounded in Ω, we obtain

$$\sum_{\substack{j \neq m \\ 1 \leq j \leq N}} \{P^{(j)}(\boldsymbol{\xi}_j) H^{(k)}(\mathbf{O}^{(j)}, \mathbf{y}) - P^{(j)}(\boldsymbol{\xi}_j) H(\mathbf{O}^{(j)}, \mathbf{O}^{(j)}) P^{(j,k)T}(\boldsymbol{\eta}_j)\} = O(\varepsilon) \, , \tag{257}$$

and

$$\sum_{j=1}^{N} \sum_{\substack{l \neq j \neq m \\ 1 \leq l \leq N}} P^{(l)}(\boldsymbol{\xi}_l) G(\mathbf{O}^{(l)}, \mathbf{O}^{(j)}) P^{(j,k)T}(\boldsymbol{\eta}_j) = O(\varepsilon) \, , \tag{258}$$

for $\mathbf{x} \in \partial\omega_\varepsilon^{(m)}, \mathbf{y} \in \Omega_\varepsilon$.

Thus combining the estimates (255)-(258) in (254), we have

$$|R_\varepsilon^{(k)}(\mathbf{x}, \mathbf{y})| \leq \text{const } \varepsilon \, , \tag{259}$$

for $\mathbf{x} \in \partial\omega_\varepsilon^{(m)}, \mathbf{y} \in \Omega_\varepsilon, 1 \leq m \leq N$.

Now we estimate the right-hand side of the boundary condition (253).

Using Lemma 9 i), we obtain

$$
\begin{aligned}
&\left|\sum_{j=1}^{N}\{P^{(j)}(\boldsymbol{\xi}_j)H^{(k)}(\mathbf{O}^{(j)},\mathbf{y})-\varepsilon H(\mathbf{x},\mathbf{O}^{(j)})B^{(j)}H^{(k)}(\mathbf{O}^{(j)},\mathbf{y})\}\right| \\
=&\left|\sum_{j=1}^{N}\{(P^{(j)}(\boldsymbol{\xi}_j)-\Gamma(\boldsymbol{\xi}_j,\mathbf{O})B^{(j)})H^{(k)}(\mathbf{O}^{(j)}\ ,\mathbf{y})\}\right| \\
\le&\ \text{const}\ \sum_{j=1}^{N}\varepsilon^2|\mathbf{x}-\mathbf{O}^{(j)}|^{-2}\le \text{const}\ \varepsilon^2\ ,\ \text{ for } \mathbf{x}\in\partial\Omega\ ,\mathbf{y}\in\partial\Omega_\varepsilon\ , \qquad (260)
\end{aligned}
$$

where we have used the fact that for $\mathbf{x}\in\partial\Omega$, $1\le|\mathbf{x}-\mathbf{O}^{(j)}|$, $1\le j\le N$.

From Lemma 9 *ii*), we also have

$$
|P^{(j,k)}(\boldsymbol{\xi}^{(j)})|\le \text{const}\ \varepsilon(|\mathbf{x}-\mathbf{O}^{(j)}|)^{-1}\ , \qquad (261)
$$

and this combined with Lemma 9 *i*) yields

$$
\begin{aligned}
&\left|\varepsilon^{-1}\sum_{j=1}^{N}\{h^{(j,k)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)-H(\mathbf{x},\mathbf{O}^{(j)})P^{(j,k)T}(\boldsymbol{\eta}_j)\}\right| \\
=&\ \varepsilon^{-1}\left|\sum_{j=1}^{N}\{h^{(j,k)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)-\Gamma(\boldsymbol{\xi}_j,\mathbf{O})P^{(j,k)T}(\boldsymbol{\eta}_j)\}\right| \\
\le&\ \text{const}\ \sum_{j=1}^{N}\varepsilon^2|\mathbf{x}-\mathbf{O}^{(j)}|^{-2}|\mathbf{y}-\mathbf{O}^{(j)}|^{-1} \\
\le&\ \text{const}\ \sum_{j=1}^{N}\varepsilon^2|\mathbf{y}-\mathbf{O}^{(j)}|^{-1}\ ,\ \text{ for } \mathbf{x}\in\partial\Omega\ ,\mathbf{y}\in\partial\Omega_\varepsilon\ . \qquad (262)
\end{aligned}
$$

We have, by (261) and the definition of G and its regular part H, the estimates

$$
|P^{(j)}(\boldsymbol{\xi}_j)H(\mathbf{O}^{(j)},\mathbf{O}^{(j)})P^{(j,k)T}(\boldsymbol{\eta}_j))|\le \text{const}\ \varepsilon^2|\mathbf{y}-\mathbf{O}^{(j)}|^{-1}\ , \qquad (263)
$$

and

$$
|P^{(l)}(\boldsymbol{\xi}_l)G(\mathbf{O}^{(l)},\mathbf{O}^{(j)})P^{(j,k)T}(\boldsymbol{\eta}_j)|\le \text{const}\ \varepsilon^2|\mathbf{y}-\mathbf{O}^{(j)}|^{-1}\ ,l\neq j\ , \qquad (264)
$$

for $\mathbf{x}\in\partial\Omega$, $\mathbf{y}\in\Omega_\varepsilon$.

Therefore, combining the estimates (260), (262), (263) and (264) we have

$$
|R_\varepsilon^{(k)}(\mathbf{x},\mathbf{y})|\le \text{const}\ \sum_{j=1}^{N}\varepsilon^2|\mathbf{y}-\mathbf{O}^{(j)}|^{-1}\ , \qquad (265)
$$

for $\mathbf{x} \in \partial\Omega$, $\mathbf{y} \in \Omega_\varepsilon$.

Then (259), (265) and Lemma 5 imply

$$
\begin{aligned}
|R_\varepsilon^{(k)}(\mathbf{x},\mathbf{y})| &\leq \text{const } \max\left\{\sum_{j=1}^{N} \varepsilon^2 |\mathbf{x}-\mathbf{O}^{(j)}|^{-1}, \sum_{j=1}^{N} \varepsilon^2 |\mathbf{y}-\mathbf{O}^{(j)}|^{-1}\right\} \\
&\leq \text{const} \sum_{j=1}^{N} \varepsilon^2 (\min\{|\mathbf{x}-\mathbf{O}^{(j)}|, |\mathbf{y}-\mathbf{O}^{(j)}|\})^{-1}. \qquad (266)
\end{aligned}
$$

The proof is complete.

□

9 Simplified asymptotic formulae for the case of three dimensions

Here we show how the asymptotic formulae (224) simplify under certain constraints on the independent variables.

Corollary 2 *a) Let* $\mathbf{x}, \mathbf{y} \in \Omega_\varepsilon \subset \mathbb{R}^3$ *such that*

$$
\min\{|\mathbf{x}-\mathbf{O}^{(j)}|, |\mathbf{y}-\mathbf{O}^{(j)}|\} > 2\,\varepsilon \textit{ for all } j = 1, \dots, N. \qquad (267)
$$

Then

$$
\begin{aligned}
G_\varepsilon(\mathbf{x},\mathbf{y}) &= G(\mathbf{x},\mathbf{y}) - \varepsilon \sum_{j=1}^{N} G(\mathbf{x},\mathbf{O}^{(j)}) B^{(j)} G(\mathbf{O}^{(j)},\mathbf{y}) \\
&+ O\left(\sum_{j=1}^{N} \varepsilon^2 (\min\{|\mathbf{x}-\mathbf{O}^{(j)}|, |\mathbf{y}-\mathbf{O}^{(j)}|\})^{-1}\right), \qquad (268)
\end{aligned}
$$

b) If $\max\{|\mathbf{x}-\mathbf{O}^{(m)}|, |\mathbf{y}-\mathbf{O}^{(m)}|\} < 1/2$, *then*

$$
\begin{aligned}
G_\varepsilon(\mathbf{x},\mathbf{y}) = \; & \varepsilon^{-1} g^{(m)}(\boldsymbol{\xi}_m, \boldsymbol{\eta}_m) \\
& -(I_3 - P^{(m)}(\boldsymbol{\xi}_m)) H(\mathbf{O}^{(m)}, \mathbf{O}^{(m)}) (I_3 - P^{(m)T}(\boldsymbol{\eta}_m)) \\
& + O(\max\{|\mathbf{x}-\mathbf{O}^{(m)}|, |\mathbf{y}-\mathbf{O}^{(m)}|\}). \qquad (269)
\end{aligned}
$$

We note that the formula (268) presented for in part a) of the above Corollary is similar to that presented in the paper by Ozawa [13] (p. 215), for the approximate Green's function of the eigenvalue problem for the Laplacian in a bounded domain in $\mathbb{R}^3$ containing several circular inclusions, which makes use of the Green's function in the unperturbed domain.

Proof. *a*) From (224), G_ε can be rewritten as

$$\begin{aligned}
G_\varepsilon(\mathbf{x},\mathbf{y}) &= G(\mathbf{x},\mathbf{y}) - \varepsilon^{-1}\sum_{j=1}^{N} h^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) \\
&+\sum_{j=1}^{N}\big\{P^{(j)}(\boldsymbol{\xi}_j)H(\mathbf{O}^{(j)},\mathbf{y}) + H(\mathbf{x},\mathbf{O}^{(j)})P^{(j)T}(\boldsymbol{\eta}_j) \\
&-P^{(j)}(\boldsymbol{\xi}_j)H(\mathbf{O}^{(j)},\mathbf{O}^{(j)})P^{(j)T}(\boldsymbol{\eta}_j) - \varepsilon H(\mathbf{x},\mathbf{O}^{(j)})B^{(j)}H(\mathbf{O}^{(j)},\mathbf{y})\big\} \\
&+\sum_{j=1}^{N}\sum_{\substack{k\neq j\\ 1\le k\le N}} P^{(k)}(\boldsymbol{\xi}_k)G(\mathbf{O}^{(k)},\mathbf{O}^{(j)})P^{(j)T}(\boldsymbol{\eta}_j) \\
&+O\left(\sum_{j=1}^{N}\varepsilon^2(\min\{|\mathbf{x}-\mathbf{O}^{(j)}|\ ,|\mathbf{y}-\mathbf{O}^{(j)}|\})^{-1}\right)\ . \qquad (270)
\end{aligned}$$

By Lemma 9, we have the following estimate for the elastic capacitary potential

$$P^{(j)}(\boldsymbol{\xi}_j) = \varepsilon\Gamma(\mathbf{x},\mathbf{O}^{(j)})B^{(j)} + O\left(\varepsilon^2|\mathbf{x}-\mathbf{O}^{(j)}|^{-2}\right)\ , \qquad (271)$$

and from Lemma 10 we also have the approximation

$$\begin{aligned}
\varepsilon^{-1}h^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) &= \Gamma(\mathbf{x},\mathbf{O}^{(j)})P^{(j)T}(\boldsymbol{\eta}_j) + O\left(\varepsilon^2(|\mathbf{x}-\mathbf{O}^{(j)}|^2|\mathbf{y}-\mathbf{O}^{(j)}|)^{-1}\right) \\
&= \varepsilon\Gamma(\mathbf{x},\mathbf{O}^{(j)})B^{(j)}\Gamma(\mathbf{y},\mathbf{O}^{(j)}) \\
&+O\left(\varepsilon^2(|\mathbf{x}-\mathbf{O}^{(j)}||\mathbf{y}-\mathbf{O}^{(j)}|\min\{|\mathbf{x}-\mathbf{O}^{(j)}|,|\mathbf{y}-\mathbf{O}^{(j)}|\})^{-1}\right)\ , \qquad (272)
\end{aligned}$$

where in (272) we have combined both of the above mentioned results.

In (270), using the (271) and (272), we have

$$\begin{aligned}
G_\varepsilon(\mathbf{x},\mathbf{y}) &= G(\mathbf{x},\mathbf{y}) - \varepsilon\sum_{j=1}^{N}\Gamma(\mathbf{x},\mathbf{O}^{(j)})B^{(j)}\Gamma(\mathbf{y},\mathbf{O}^{(j)}) \\
&+\sum_{j=1}^{N}\big\{\varepsilon\Gamma(\mathbf{x},\mathbf{O}^{(j)})B^{(j)}H(\mathbf{O}^{(j)},\mathbf{y}) + \varepsilon H(\mathbf{x},\mathbf{O}^{(j)})B^{(j)}\Gamma(\mathbf{y},\mathbf{O}^{(j)}) \\
&-\varepsilon H(\mathbf{x},\mathbf{O}^{(j)})B^{(j)}H(\mathbf{O}^{(j)},\mathbf{y})\big\} \\
&+O\left(\sum_{j=1}^{N}\varepsilon^2(\min\{|\mathbf{x}-\mathbf{O}^{(j)}|,|\mathbf{y}-\mathbf{O}^{(j)}|\})^{-1}\right)\ . \qquad (273)
\end{aligned}$$

Using the definition of the matrix function G given in (215), we may rewrite the preceding formula as

$$\begin{aligned} G_\varepsilon(\mathbf{x},\mathbf{y}) &= G(\mathbf{x},\mathbf{y}) - \varepsilon \sum_{j=1}^N G(\mathbf{x},\mathbf{O}^{(j)}) B^{(j)} \Gamma(\mathbf{y},\mathbf{O}^{(j)}) \\ &\quad + \varepsilon \sum_{j=1}^N G(\mathbf{x},\mathbf{O}^{(j)}) B^{(j)} H(\mathbf{O}^{(j)},\mathbf{y}) \\ &+ O\left(\sum_{j=1}^N \varepsilon^2 (\min\{|\mathbf{x}-\mathbf{O}^{(j)}|, |\mathbf{y}-\mathbf{O}^{(j)}|\})^{-1} \right) , \end{aligned} \tag{274}$$

from which (268) follows.

b) Due to the condition $\max\{|\mathbf{x}-\mathbf{O}^{(m)}|, |\mathbf{y}-\mathbf{O}^{(m)}|\} < 1/2$, and since $H(\mathbf{x},\mathbf{y})$ has smooth components for $\mathbf{x}$, $\mathbf{y} \in \Omega$, in the vicinity of $(\mathbf{O}^{(m)},\mathbf{O}^{(m)})$ we have from (224)

$$\begin{aligned} G_\varepsilon(\mathbf{x},\mathbf{y}) &= -H(\mathbf{O}^{(m)},\mathbf{O}^{(m)}) + \varepsilon^{-1} \sum_{j=1}^N g^{(j)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) - (N-1)\Gamma(\mathbf{x},\mathbf{y}) \\ &\quad + P^{(m)}(\boldsymbol{\xi}_m)(H(\mathbf{O}^{(m)},\mathbf{O}^{(m)}) + O(|\mathbf{y}-\mathbf{O}^{(m)}|)) \\ &\quad + (H(\mathbf{O}^{(m)},\mathbf{O}^{(m)}) + O(|\mathbf{x}-\mathbf{O}^{(m)}|)) P^{(m)T}(\boldsymbol{\eta}_m) \\ &\quad - P^{(m)}(\boldsymbol{\xi}_m) H(\mathbf{O}^{(m)},\mathbf{O}^{(m)}) P^{(m)T}(\boldsymbol{\eta}_m) \\ &\quad + \sum_{\substack{j\neq m \\ 1\le j\le N}} \big\{ P^{(j)}(\boldsymbol{\xi}_j)(H(\mathbf{O}^{(j)},\mathbf{O}^{(m)}) + O(|\mathbf{y}-\mathbf{O}^{(m)}|)) \\ &\quad + (H(\mathbf{O}^{(m)},\mathbf{O}^{(j)}) + O(|\mathbf{x}-\mathbf{O}^{(m)}|)) P^{(j)T}(\boldsymbol{\eta}_j) \\ &\quad - P^{(j)}(\boldsymbol{\xi}_j) H(\mathbf{O}^{(j)},\mathbf{O}^{(j)}) P^{(j)T}(\boldsymbol{\eta}_j) \big\} \\ &\quad + \sum_{j=1}^N \sum_{\substack{k\neq j \\ 1\le k\le N}} P^{(k)}(\boldsymbol{\xi}_k) G(\mathbf{O}^{(k)},\mathbf{O}^{(j)}) P^{(j)T}(\boldsymbol{\eta}_j) , \\ &\quad + O(\max\{|\mathbf{x}-\mathbf{O}^{(m)}|, |\mathbf{y}-\mathbf{O}^{(m)}|\}) \end{aligned} \tag{275}$$

Now using the estimate for the regular part $h^{(j)}$ given in (272), and that for the elastic capacitary potential (271) for $j \neq m$ we arrive at (269).

□

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