

On the Brezis and Mironescu conjecture concerning a Gagliardo-Nirenberg inequality for fractional Sobolev norms

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Abstract

We prove the Gagliardo-Nirenberg type inequality

$$\|u\|_{W^{\theta s, p/\theta}} \leq c(n) \left(\frac{p}{p-1}\right)^\theta \left(\frac{1-s}{1-\theta}\right)^{\theta/p} \|u\|_{W^{s, p}}^\theta \|u\|_{L^\infty}^{1-\theta},$$

where $0 < \theta < 1$, $0 < s < 1$, $1 < p < \infty$, and $\|u\|_{W^{s, p}}$ is the seminorm in the fractional Sobolev space $W^{s, p}(\mathbf{R}^n)$. The dependence of the constant factor in the right-hand side on each of the parameters s , θ , and p is precise in a sense.

Let $s \in (0, 1)$ and let $1 < p \leq \infty$. We introduce the space $W^{s, p}(\mathbf{R}^n)$ of functions in \mathbf{R}^n with the finite seminorm

$$\|u\|_{W^{s, p}} = \left(\int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy \right)^{1/p}.$$

Recently Bourgain, Brezis and Mironescu [1] found the relation

$$\lim_{s \uparrow 1} (1-s)^{1/p} \|u\|_{W^{s, p}} = c \|u\|_{W^{1, p}}, \tag{1}$$

which subsequently motivated Brezis and Mironescu to conjecture the Gagliardo-Nirenberg type inequality

$$\|u\|_{W^{s/2, 2p}} \leq c(n, p) (1-s)^{1/2p} \|u\|_{W^{s, p}}^{1/2} \|u\|_{L^\infty}^{1/2} \tag{2}$$

(see [2], Remark 5). In [2] one can also read: ‘‘It would be of interest to establish

$$\|u\|_{W^{\theta s, p/\theta}} \leq c \|u\|_{W^{s, p}}^\theta \|u\|_{L^\infty}^{1-\theta}, \quad 0 < \theta < 1, \tag{3}$$

with control of the constant c , in particular when $s \uparrow 1$ ’.

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In the present paper we prove that (3) holds with $c = c(n, p, \theta)(1 - s)^{\theta/p}$, which, obviously, contains inequality (2) predicted by Brezis and Mironescu. Our proof is straightforward and rather elementary. In concluding Remarks 1 and 2 we show that the dependence of c on each of the parameters s , θ , and p is sharp in a certain sense.

Theorem. *For all $u \in W^{s,p} \cap L^\infty$ there holds the inequality*

$$\|u\|_{W^{\theta s, p/\theta}} \leq c(n) \left(\frac{p}{p-1}\right)^\theta \left(\frac{1-s}{1-\theta}\right)^{\theta/p} \|u\|_{W^{s,p}}^\theta \|u\|_{L^\infty}^{1-\theta}, \quad (4)$$

where $0 < s < 1$, $1 < p < \infty$, and $0 < \theta < 1$.

Proof. Clearly,

$$\|\bar{u}\|_{W^{\theta s, p/\theta}} \leq \max\{2^{\theta/p}, 2^{1-\theta}\} \|u\|_{L^\infty}^{1-\theta} \|u\|_{W^{s,p}}^\theta. \quad (5)$$

Hence it suffices to prove (4) only for $s \geq 1/2$.

Let $B_r(x) = \{\xi \in \mathbf{R}^n : |\xi - x| < r\}$ and $B_r(0) = B_r$. We introduce the mean value $\bar{u}_{x,y}$ of u over the ball $\mathcal{B}_{x,y} := B_{|x-y|/2}((x+y)/2)$. Since

$$|u(x) - u(y)|^{p/\theta} \leq 2^{-1+p/\theta} (|u(x) - \bar{u}_{x,y}|^{p/\theta} + |\bar{u}_{x,y} - u(y)|^{p/\theta}),$$

it follows that

$$\|u\|_{W^{\theta s, p/\theta}} \leq 2 \left(\int_{\mathbf{R}^n} D(x)^{p/\theta} dx \right)^{\theta/p}, \quad (6)$$

where

$$D(x) = \left(\int_{\mathbf{R}^n} \frac{|u(x) - \bar{u}_{x,y}|^{p/\theta}}{|x-y|^{n+ps}} dy \right)^{\theta/p}.$$

We note that

$$\int_{|x-y|>\delta} \frac{|u(x) - \bar{u}_{x,y}|^{p/\theta}}{|x-y|^{n+ps}} dy \leq \frac{2^{p/\theta} |\partial B_1|}{ps} \|u\|_{L^\infty}^{p/\theta} \delta^{-ps}, \quad (7)$$

where $|\partial B_1|$ is the area of the unit sphere.

Let U be an arbitrary extension of u onto $\mathbf{R}_+^{n+1} = \{(x, z) : x \in \mathbf{R}^n, z > 0\}$ such that $\nabla U \in L_{loc}^1(\mathbf{R}_+^{n+1})$. By $\bar{U}_{x,y}(z)$ we denote the mean value of $U(\cdot, z)$ in $\mathcal{B}_{x,y}$. Using the identity

$$|x-y|^{(1-s)p} = p(1-s) \int_0^{|x-y|} z^{-1+p(1-s)} dz,$$

we find

$$\begin{aligned} & \int_{|x-y|<\delta} \frac{|u(x) - \bar{u}_{x,y}|^{p/\theta}}{|x-y|^{n+ps}} dy = \\ & p^{1/\theta} (1-s)^{1/\theta} \int_{|x-y|<\delta} \left(\int_0^{|x-y|} z^{-1+p(1-s)} |u(x) - \bar{u}_{x,y}|^p dz \right)^{1/\theta} \frac{dy}{|x-y|^{n+ps+(1-s)p/\theta}} \leq \\ & 3^{-1+p/\theta} p^{1/\theta} (1-s)^{1/\theta} (\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3), \end{aligned} \quad (8)$$

where

$$\mathcal{J}_1 := \int_{|x-y|<\delta} \left(\int_0^{|x-y|} z^{-1+p(1-s)} |u(x) - U(x, z)|^p dz \right)^{1/\theta} \frac{dy}{|x-y|^{n+ps+(1-s)p/\theta}},$$

$$\mathcal{J}_2 := \int_{|x-y|<\delta} \left(\int_0^{|x-y|} z^{-1+p(1-s)} |\bar{U}_{x,y}(z) - \bar{u}_{x,y}|^p dz \right)^{1/\theta} \frac{dy}{|x-y|^{n+ps+(1-s)p/\theta}},$$

and

$$\mathcal{J}_3 := \int_{|x-y|<\delta} \left(\int_0^{|x-y|} z^{-1+p(1-s)} |U(x, z) - \bar{U}_{x,y}(z)|^p dz \right)^{1/\theta} \frac{dy}{|x-y|^{n+ps+(1-s)p/\theta}}.$$

Clearly,

$$\begin{aligned} \mathcal{J}_1 &\leq \int_{|x-y|<\delta} \left(\int_0^{|x-y|} z^{-1+p(1-s)} \left(\int_0^z \left| \frac{\partial U(x, t)}{\partial t} \right| dt \right)^p dz \right)^{1/\theta} \frac{dy}{|x-y|^{n+ps+(1-s)p/\theta}} \leq \\ &\int_{|x-y|<\delta} \left(\int_0^{|x-y|} z^{-1-ps} \left(\int_0^z \left| \frac{\partial U(x, t)}{\partial t} \right| dt \right)^p dz \right)^{1/\theta} \frac{dy}{|x-y|^{n-ps(1-\theta)/\theta}}. \end{aligned}$$

By Hardy's inequality

$$\int_0^a z^{-1-sp} \left| \int_0^z \varphi(t) dt \right|^p dz \leq s^{-p} \int_0^a z^{-1+p(1-s)} |\varphi(z)|^p dz,$$

one has

$$\begin{aligned} \mathcal{J}_1 &\leq s^{-p/\theta} \int_{|x-y|<\delta} \left(\int_0^{|x-y|} z^{-1+p(1-s)} \left| \frac{\partial U(x, z)}{\partial z} \right|^p dz \right)^{1/\theta} \frac{dy}{|x-y|^{n-ps(1-\theta)/\theta}} \leq \\ &\frac{\theta |\partial B_1|}{s^{p/\theta} ps(1-\theta)} \left(\int_0^\infty z^{-1+p(1-s)} \left| \frac{\partial U(x, z)}{\partial z} \right|^p dz \right)^{1/\theta} \delta^{ps(1-\theta)/\theta}. \end{aligned} \quad (9)$$

Duplicating the same argument, we conclude that

$$\mathcal{J}_2 \leq s^{-p/\theta} \int_{|x-y|<\delta} \frac{dy}{|x-y|^{n-ps(1-\theta)/\theta}} \left(\int_0^{|x-y|} z^{-1+p(1-s)} \left| \frac{\partial \bar{U}_{x,y}(z)}{\partial z} \right|^p dz \right)^{1/\theta}. \quad (10)$$

Let \mathcal{M} denote the n -dimensional Hardy-Littlewood maximal operator

$$(\mathcal{M}f)(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r(x)} |f(\xi)| d\xi.$$

Using the obvious inequality

$$\left| \frac{\partial \bar{U}_{x,y}(z)}{\partial z} \right| \leq \left(\mathcal{M} \frac{\partial U}{\partial z} \right)(x, z),$$

we find from (10)

$$\mathcal{J}_2 \leq \frac{\theta |\partial B_1|}{s^{p/\theta} ps(1-\theta)} \left(\int_0^\infty z^{-1+p(1-s)} \left(\mathcal{M} \frac{\partial U}{\partial z} \right)^p dz \right)^{1/\theta} \delta^{ps(1-\theta)/\theta}. \quad (11)$$

In order to estimate \mathcal{J}_3 we use the Sobolev type integral representation in the form given in [3], Ch. 10, Sect. 3

$$U(x, z) - \bar{U}_{x,y}(z) = \sum_{k=1}^n \int_{\mathcal{B}_{x,y}} \frac{b_k(\xi, x)}{|x - \xi|^{n-1}} \frac{\partial U(\xi, z)}{\partial \xi_k} d\xi, \quad (12)$$

where $b_k(\xi, x)$ are continuous functions for $x \neq \xi$ admitting the estimate

$$|b_k(\xi, x)| \leq \frac{|x - y|^n}{n|\mathcal{B}_{x,y}|}.$$

Clearly, (12) implies the estimate

$$|U(x, z) - \bar{U}_{x,y}(z)| \leq \frac{2^n n^{1/2}}{|\partial B_1|} \int_{B_r(x)} \frac{|\nabla_\xi U(\xi, z)|}{|x - \xi|^{n-1}} d\xi,$$

where $r = |x - y|$. Integrating by parts we find

$$\begin{aligned} \int_{B_r(x)} \frac{|\nabla_\xi U(\xi, z)|}{|x - \xi|^{n-1}} d\xi &= r^{1-n} \int_{B_r(x)} |\nabla_\xi U(\xi, z)| d\xi + \\ (n-1) \int_0^r \frac{ds}{s^n} \int_{B_s(x)} |\nabla_\xi U(\xi, z)| d\xi &\leq n|x - y|(\mathcal{M}|\nabla U|)(x, z). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{J}_3 &\leq \left(\frac{2^n n^{3/2}}{|\partial B_1|} \right)^{p/\theta} \int_{|x-y|<\delta} \left(\int_0^{|x-y|} z^{-1+p(1-s)} (\mathcal{M}|\nabla U|)^p dz \right)^{1/\theta} \frac{dy}{|x-y|^{n-ps(1-\theta)/\theta}} \leq \\ &\frac{(2^n n^{3/2})^{p/\theta\theta}}{|\partial B_1|^{(p-\theta)/\theta} p s(1-\theta)} \left(\int_0^\infty z^{-1+p(1-s)} (\mathcal{M}|\nabla U|)^p dz \right)^{1/\theta} \delta^{ps(1-\theta)/\theta}. \end{aligned} \quad (13)$$

Here and in the sequel, for the sake of brevity, by $\mathcal{M}|\nabla U|$ we mean $(\mathcal{M}|\nabla U|)(x, z)$. Putting estimates (9), (11), and (13) into (8), we arrive at

$$\int_{|x-y|<\delta} \frac{|u(x) - \bar{u}_{x,y}|^{p/\theta}}{|x-y|^{n+ps}} dy \leq c(n) \frac{(1-s)^{1/\theta}}{1-\theta} \left(\int_0^\infty z^{-1+p(1-s)} (\mathcal{M}|\nabla U|)^p dz \right)^{1/\theta} \delta^{ps(1-\theta)/\theta}.$$

This estimate together with (7) implies that $D(x)$ is majorized by

$$c(n) \left(\|u\|_{L^\infty} \delta^{-\theta s} + \left(\frac{1-s}{1-\theta} \right)^{1/p} \left(\int_0^\infty z^{-1+p(1-s)} (\mathcal{M}|\nabla U|)^p dz \right)^{1/p} \delta^{s(1-\theta)} \right).$$

Minimizing the right-hand side, we conclude that

$$D(x) \leq c(n) \left(\frac{1-s}{1-\theta} \right)^{\theta/p} \|u\|_{L^\infty}^{1-\theta} \left(\int_0^\infty z^{-1+p(1-s)} (\mathcal{M}|\nabla U|)^p dz \right)^{\theta/p}.$$

Hence and by (6)

$$\|u\|_{W^{\theta s, p/\theta}} \leq c(n) \left(\frac{1-s}{1-\theta} \right)^{\theta/p} \|u\|_{L^\infty}^{1-\theta} \left(\int_{\mathbf{R}^n} \int_0^\infty z^{-1+p(1-s)} (\mathcal{M}|\nabla U|)^p dz dx \right)^{\theta/p}.$$

Since

$$\|\mathcal{M}u\|_{L^p} \leq \frac{n8^n p}{|\partial B_1|(p-1)} \|u\|_{L^p}$$

(see [4], Sect. 2.5), we have

$$\|u\|_{W^{\theta s, p/\theta}} \leq c(n) \left(\frac{p}{p-1}\right)^\theta \left(\frac{1-s}{1-\theta} \int_0^\infty \int_{\mathbf{R}^n} z^{-1+p(1-s)} |\nabla U(x, z)|^p dx dz\right)^{\theta/p} \|u\|_{L^\infty}^{1-\theta}. \quad (14)$$

Now we define U by the formula

$$U(x, z) := \int_{\mathbf{R}^n} \psi(h) u(x + zh) dh, \quad (15)$$

where

$$\psi(h) = |\partial B_1| n(n+1)(1-|h|)_+$$

with plus standing for the nonnegative part of a real valued function. It follows directly from (15) that

$$|\nabla U(x, z)| \leq \frac{n(n+1)(n+2)}{z |\partial B_1|} \int_{|h|<1} |u(x+zh) - u(x)| dh.$$

Hence and by Hölder's inequality

$$\begin{aligned} & \int_0^\infty \int_{\mathbf{R}^n} z^{-1+p(1-s)} |\nabla U(x, z)|^p dx dz \leq \\ & \frac{n}{|\partial B_1|} (n+1)^p (n+2)^p \int_0^\infty z^{-1-ps} \int_{|h|<1} \int_{\mathbf{R}^n} |u(x+zh) - u(x)|^p dx dh dz. \end{aligned} \quad (16)$$

We have

$$\begin{aligned} & \int_0^\infty z^{-1-ps} \int_{|h|<1} |u(x+zh) - u(x)|^p dh dz = \\ & \int_0^\infty z^{-1-ps-n} \int_0^z \rho^{n-1} d\rho \int_{\partial B_1} |u(x+\rho\theta) - u(x)|^p d\theta = \\ & (ps+n)^{-1} \int_0^\infty \rho^{-ps-1} d\rho \int_{\partial B_1} |u(x+\rho\theta) - u(x)|^p d\theta. \end{aligned}$$

Thus,

$$\int_0^\infty \int_{\mathbf{R}^n} z^{-1+p(1-s)} |\nabla U(x, z)|^p dx dz \leq \frac{n(n+1)^p (n+2)^p}{|\partial B_1|(ps+n)} \|u\|_{W^{s,p}}^p. \quad (17)$$

Combining (17) with (14) we complete the proof.

Remark 1. Let

$$\|u\|_{W^{1,p}} = \left(\int_{\mathbf{R}^n} |\nabla u(x)|^p dx \right)^{1/p}.$$

As a particular case of a more general inequality, Brezis and Mironescu [2] obtained (3) for $s = 1$. They commented on this in the following way: “We do not know any elementary (i.e., without the Littlewood-Paley machinery) proof of (3) when $s = 1$ ”.

Obviously, the above proof of (4), complemented by the reference to formula (1), provides an elementary proof of the inequality

$$(1 - \theta)^{\theta/p} \|u\|_{W^{\theta,p/\theta}} \leq c(n, p) \|u\|_{W^{1,p}}^{\theta} \|u\|_{L^{\infty}}^{1-\theta}.$$

The factor $(1 - \theta)^{\theta/p}$ controls the blow up of the norm in $W^{\theta,p/\theta}$ as $\theta \uparrow 1$.

Remark 2. Note that passing to the limit as $p \rightarrow \infty$ in both sides of (4) one obtains inequality (3) with $p = \infty$ and with a certain finite constant c . Let us consider the case $p \rightarrow 1$ when the constant factor in (4) tends to infinity. It follows from (4) that the best value of $c(n, p, \theta)$ in the inequality

$$\|u\|_{W^{\theta s, p/\theta}} \leq c(n, p, \theta) (1 - s)^{\theta/p} \|u\|_{W^{s,p}}^{\theta} \|u\|_{L^{\infty}}^{1-\theta} \quad (18)$$

admits the upper estimate

$$\limsup_{p \downarrow 1} (p - 1)^{\theta} c(n, p, \theta) \leq c(n) (1 - \theta)^{-\theta}. \quad (19)$$

Now we obtain the analogous lower estimate

$$\liminf_{p \downarrow 1} (p - 1)^{\theta} c(n, p, \theta) \geq 1. \quad (20)$$

In fact, the characteristic function χ of the ball B_1 belongs to $W^{s,p}$ and $W^{\theta s, p/\theta}$ if and only if $sp < 1$, and there holds

$$\|\chi\|_{W^{\theta s, p/\theta}} = \|\chi\|_{W^{s,p}}^{\theta}.$$

Putting $u = \chi$ into (18), where $s = p^{-1} - \varepsilon$ with an arbitrarily small $\varepsilon > 0$, we obtain

$$1 \leq c(n, p, \theta) ((p - 1)/p)^{\theta/p},$$

which implies (20). Thus, the growth $O((p - 1)^{-\theta})$ of the constant in (4) as $p \downarrow 1$ is best possible.

References

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