PATHOLOGICAL SOLUTIONS TO ELLIPTIC PROBLEMS IN DIVERGENCE FORM WITH CONTINUOUS COEFFICIENTS

TIANLING JIN, VLADIMIR MAZ'YA, AND JEAN VAN SCHAFTINGEN

RÉSUMÉ. We construct a function $u \in W^{1,1}_{\mathrm{loc}}(B(0,1))$ which is a solution to $\operatorname{div}(A\nabla u) = 0$ in the sense of distributions, where A is continuous and $u \notin W^{1,p}_{\mathrm{loc}}(B(0,1))$ for $p > 1$. We also give a function $u \in W^{1,1}_{\mathrm{loc}}(B(0,1))$ such that $u \in W^{1,p}_{\mathrm{loc}}(B(0,1))$ for every $p < \infty$, u satisfies $\operatorname{div}(A\nabla u) = 0$ with A continuous but $u \notin W^{1,\infty}_{\mathrm{loc}}(B(0,1))$.

1. Introduction

Consider the equation

$$-\operatorname{div} A\nabla u = 0 \qquad \text{in } \Omega, \tag{1}$$

for $\Omega \subset \mathbf{R}^n$. If $A : \Omega \to \mathbf{R}^{n\times n}$ is bounded, measurable and elliptic, i.e., there exists $\lambda, \Lambda \in \mathbf{R}_*$ such that for every $x \in \Omega$, $A(x)$ is a symmetric matrix, and

$$|\xi|^2 \le (A(x)\xi)\cdot\xi \le \Lambda|\xi|^2,$$

then one can define a weak solution $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ by requiring that for every $\varphi \in C^1_c(\Omega)$,

$$\int_\Omega (A\nabla u)\cdot\nabla\varphi = 0.$$

We are interested in the regularity properties of u. A fundamental result of E. De Giorgi [3] states that if $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$, then u is locally Hölder continuous. In particular, u is then locally bounded. In the same direction, N. G. Meyers [8] also proved that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ for some $p > 2$.

J. Serrin [9] showed that the assumption $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is essential in E. De Giorgi's result by constructing for every $p \in (1,2)$ a function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ that solves such an elliptic equation but which is not locally bounded. In these counterexamples A is not continuous. J. Serrin [9] conjectured that if A was Hölder continuous, then any weak solution $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is in $W^{1,2}_{\mathrm{loc}}(\Omega)$, and one can then apply E. De Giorgi's theory.

This conjectured was confirmed for $u \in W^{1,p}(\Omega)$ by R. A. Hager and J. Ross [4] and for $u \in W^{1,1}(\Omega)$ by H. Brezis [1, 2]. The proof of Brezis extends to the case where the modulus of continuity of A

$$\omega_A(t) = \sup_{\substack{x,y\in\Omega\\|x-y|\le t}} |A(x) - A(y)|, \tag{2}$$

satisfies the Dini condition

$$\int_0^1 \frac{\omega_A(s)}{s}\,ds < \infty. \tag{3}$$

In the case where A is merely continuous, H. Brezis obtained the following result

Theorem 1.1 (H. Brezis [1, 2]). *Assume that $A \in C(\Omega; \mathbf{R}^{n\times n})$ is elliptic. If $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ solves (1), then for every $q \in [p, +\infty)$, one has $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$.*

H. Brezis asked two questions about the cases $p = 1$ and $q = \infty$ in the previous theorem. We answer both questions, with a negative answer. First we have

Proposition 1.2. *There exists $u \in W^{1,1}_{\mathrm{loc}}(B(0,1))$ and an elliptic $A \in C(B(0,1); \mathbf{R}^{n\times n})$ such that u solves (1), but $u \notin W^{1,p}_{\mathrm{loc}}(B(0,1))$ for every $p > 1$.*

As a byproduct, we obtain

Proposition 1.3. *There exists $A \in C(B(0,1); \mathbf{R}^{n\times n})$ such that the problem*

$$\begin{cases} -\operatorname{div}(A\nabla u) = 0 & \text{in } B(0,1), \\ u = 0 & \text{on } \partial B(0,1). \end{cases} \tag{4}$$

has a nontrivial solution.

Our construction in Proposition 1.2 allows in fact to show that the counterexamples can be improved

Proposition 1.4. *There exists $u \in W^{1,1}_{\mathrm{loc}}(B(0,1))$ and an elliptic $A \in C(B(0,1); \mathbf{R}^{n\times n})$ such that u solves (1), $Du \in (L \log L)_{\mathrm{loc}}(B(0,1))$ but $u \notin W^{1,p}_{\mathrm{loc}}(B(0,1))$ for every $p > 1$.*

In particular, in this case, Du belongs locally to the Hardy space $\mathcal{H}^1$ (see [10]).

Concerning the possibility of Lipschitz estimates, we have

Proposition 1.5. *There exists $u \in W^{1,1}_{\mathrm{loc}}(B(0,1))$ and an elliptic $A \in C(B(0,1); \mathbf{R}^{n\times n})$ such that u solves (1), $Du \in W^{1,p}_{\mathrm{loc}}(B(0,1))$ for every $p > 1$, $Du \in \mathrm{BMO}_{\mathrm{loc}}(B(0,1))$ but $u \notin W^{1,\infty}_{\mathrm{loc}}((B(0,1))$.*

This shows that $Du \in L^p(B(0,1))$ does not imply $Du \in L^\infty(B(0,1/2))$, one can wonder whether it implies that $Du \in \mathrm{BMO}(B(0,1/2))$. The answer is still negative

Proposition 1.6. *There exists $u \in W^{1,1}_{\mathrm{loc}}(B(0,1))$ and an elliptic $A \in C(B(0,1); \mathbf{R}^{n\times n})$ such that u solves (1), $u \in W^{1,p}_{\mathrm{loc}}(B(0,1))$ for every $p \in (1,\infty)$ but $Du \notin \mathrm{BMO}_{\mathrm{loc}}(B(0,1))$.*

The construction of the counterexamples are made by explicit formulas, inspired by the construction of J. Serrin [9]. They can also be obtained from asymptotic formulas of V. Kozlov and V. Maz'ya [6, 7].

2. The pathological solutions

Our counterexamples rely on the following computation

Lemma 2.1. *Let $v \in C^2((0,R))$ and $\alpha \in C^1((0,R))$. Define $A(x) = (a_{ij}(x))_{\substack{1\le i\le n\\ 1\le j\le n}}$ by*

$$a_{ij}(x) = \delta_{ij} + \alpha(|x|)\Big(\delta_{ij} - \frac{x_i x_j}{|x|^2}\Big).$$

Then for every $x \in B(0,R) \setminus \{0\}$,

(5)

$$\operatorname{div}\big(A(x)\nabla(x_1 v(|x|))\big) = x_1\Big(v''(|x|) + \frac{n+1}{|x|}v'(|x|) - \frac{n-1}{|x|^2}\alpha(|x|)v(|x|)\Big).$$

Remark 1. *If P is a homogeneous harmonic polynomial of degree k, the formula generalizes to*

$$\begin{aligned}\operatorname{div}\big(A(x)\nabla(P(x)v(|x|))\big)\\ = P(x)\Big(v''(|x|) + \frac{n+2k-1}{|x|}v'(|x|) - \frac{k(n+k-2)}{|x|^2}\alpha(|x|)v(|x|)\Big).\end{aligned} \tag{6}$$

Proof of Proposition 1.2. Choose $\beta > 1$, and define for some $r_0 > 1$, for $r \in (0,1)$,

$$v(r) = \frac{1}{r^n(\log\frac{r_0}{r})^\beta}. \tag{7}$$

One takes then

$$\alpha(r) = \frac{r^2 v''(r) + (n+1)rv'(r)}{(n-1)v(r)} = \frac{-\beta n}{(n-1)\log\frac{r_0}{r}} + \frac{\beta(\beta+1)}{(n-1)\big(\log\frac{r_0}{r}\big)^2}. \tag{8}$$

One can take r_0 large enough so that $\alpha \ge -\frac{1}{2}$ on $(0,1)$; the coefficient matrix A is then uniformly elliptic. Define now $u(x) = x_1 v(|x|)$. One checks that $u \in W^{1,1}(B(0,1))$ and that u is a weak solution of (1). Indeed, it is a classical solution on $B(0,1) \setminus \{0\}$ by the previous lemma. Taking, $\varphi \in C^1_c(B(0,1))$ and $\rho \in (0,1)$, and integrating by parts we obtain

$$\begin{aligned}\int_{B(0,1)\setminus B(0,\rho)} \nabla\varphi \cdot (A\nabla u) &= -\int_{\partial B(0,\rho)} \varphi\nabla u \cdot (A\frac{x}{\rho})\\ &= -\int_{\partial B(0,\rho)} \varphi\nabla u \cdot \frac{x}{\rho}\\ &= -\int_{\partial B(0,\rho)} \varphi x_1\Big(\frac{v(\rho)}{\rho} + v'(\rho)\Big)\\ &= -\int_{\partial B(0,\rho)} (\varphi(x)-\varphi(0)) x_1\Big(\frac{v(\rho)}{\rho} + v'(\rho)\Big).\end{aligned}$$

Since $\varphi \in C^1_c(B(0,1))$, one has

$$\Big|\int_{B(0,1)\setminus B(0,\rho)} \nabla\varphi \cdot (A\nabla v)\Big| \le C\rho^n(|v(\rho)| + \rho|v'(\rho)|),$$

since the right-hand side goes to 0 as $\rho \to 0$, u is a weak solution. ☐

Remark 2. *The examples constructed in the case of merely measurable coefficients by J. Serrin* [9] *to show that a solution* $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ *need not be in* $W^{1,2}_{\mathrm{loc}}(\Omega)$ *and by N. G. Meyers* [8] *to show that for every* $p > 2$, *that a solution in* $W^{1,2}_{\mathrm{loc}}(\Omega)$ *need not be in* $W^{1,p}_{\mathrm{loc}}(\Omega)$ *can be recovered with the same construction, by taking* $v(r) = r^{\alpha}$. *The ellipticity condition requires* $\alpha < n-1$ *or* $\alpha > 1$. *This covers all the cases when* $n = 2$ *; a descent argument finishes the construction in higher dimension.*

Proof of Proposition 1.4. One checks that the counterexample constructed in the proof of Proposition 1.4 satisfies $Du \in L \log L(B(0,1))$ when $\beta > 2$. □

Similar examples can be obtained following the results of V. Kozlov and V. Maz'ya [7]. By (4) therein, if $A \in C(B(0,1); \mathbf{R}^{n\times n})$, $A(Rx) = RA(x)R$ where R is the reflection with respect to the x_1 variable and A satisfies some regularity assumptions, then the equation $-\operatorname{div}(A\nabla u) = 0$ has a solution that is odd with respect to the x_1 variable and that behaves like

$$\frac{x_1}{|x|^n} \exp\Bigl(\int_{B(0,1)\setminus B(0,|x|)} \mathcal{R}(y)\, dy\Bigr)$$

around 0, where $\mathcal{R}$ is defined following [7, (3)][1]

(9)
$$\mathcal{R}(x) = \frac{(e_1 \cdot (A(x) - A(0))e_1)(x \cdot A(0)^{-1}x) - n(e_1 \cdot (A(x) - A(0))A(0)^{-1}x)(e_1 \cdot x)}{|\partial B(0,1)| |\det A(0)|^{\frac{1}{2}} (x \cdot A(0)^{-1}x)^{\frac{n}{2}+1}}.$$

Taking A as in Lemma 2.1 with $\lim_{r\to 0} \alpha(r) = 0$, one has $\mathcal{R}(x) = \alpha(|x|)(|x|^2 - {x_1}^2)/(|\partial B(0,1)||x|^{n+2})$. Therefore, there is a solution that behaves like

$$\frac{x_1}{|x|^n} \exp\bigl(\frac{n-1}{n} \int_{|x|}^1 \alpha(r)\, \frac{dr}{r}\bigr).$$

In particular, if one takes $\alpha(r) = -\beta n/((n-1)\log \frac{r_0}{r})$, one obtains a solution that behaves like $\frac{x_1}{|x|^n}(\log \frac{r_0}{r})^{-\beta}$. One could also take $a_{ij}(x) = \delta_{ij} + \kappa(|x|)(\delta_{ij} - n\delta_{i1}\delta_{j1}\frac{{x_1}^2}{|x|^2})$ and continue the computations with now $\mathcal{R}(x) = \kappa(|x|)(|x|^2 - n{x_1}^2)^2/(|\partial B(0,1)||x|^{n+2})$.

Proof of Proposition 1.3. Let u be given by the proof of Proposition 1.2. Notice that u is smooth on $\partial B(0,1)$. Since A is bounded and elliptic, the problem

$$\begin{cases} -\operatorname{div}(A\nabla w) = 0 & \text{in } B(0,1), \\ w = u & \text{on } \partial B(0,1). \end{cases}$$

has a unique solution in $w \in W^{1,2}(B(0,1))$. Since $u \notin W^{1,2}(B(0,1))$, $u \neq w$. Hence, $u - w \in W^{1,1}(B(0,1))$ is a nontrivial solution of (4). □

Proof of Proposition 1.5. Take for $r \in (0,1)$,

(10)
$$v(r) = \log \frac{r_0}{r}$$

[1]The reader should correct the misprint in [7, (3)] and read $|S^{n-1}_{+}|$ instead of $|S^{n-1}|$.

and

$$\alpha(r) = \frac{1-(n+1)}{(n-1)\log\frac{r_0}{r}} = \frac{-n}{(n-1)\log\frac{r_0}{r}}, \tag{11}$$

where r_0 is chosen so that $\alpha(r) > -\frac{1}{2}$ on $(0,1)$. Defining $u(x) = x_1 v(|x|)$, one checks that $Du \in W^{1,p}_{\mathrm{loc}}(B(0,1))$, $Du \in \mathrm{BMO}(B(0,1))$, $u \notin W^{1,\infty}(B(0,1))$ and that u solves (1) in the sense of distributions. □

As for the previous singular pathological solutions, similar examples can be obtained from results of V. Kozlov and V. Maz'ya for solutions [6]. By (4) therein if $A \in C(B(0,1); \mathbf{R}^{n\times n})$, $A(Rx) = RA(x)R$ where R is the reflection with respect to the x_1 variable and A satisfies some regularity assumptions, then the equation $-\operatorname{div}(A\nabla u) = 0$ has a solution in $W^{1,2}(B(0,1))$ that is odd with respect to to the x_1 variable and that behaves like

$$x_1 \exp\Bigl(-\int_{B(0,1)\setminus B(0,|x|)} \mathcal{R}(y)\,dy\Bigr)$$

around 0, where $\mathcal{R}$ is given by (9). Taking A as in Lemma 2.1 with $\alpha(r) = \frac{-n}{n-1}(\log\frac{r_0}{r})^{-1}$ one recovers the counterexample presented above.

Proof of Proposition 1.6. Define for $r \in (0,1)$,

$$v(r) = \bigl(\log\frac{r_0}{r}\bigr)^2.$$

and

$$\alpha(r) = \frac{-2n}{(n-1)\log\frac{r_0}{r}} + \frac{2}{(n-1)(\log\frac{r_0}{r})^2}.$$

Defining $u(x) = x_1 v(|x|)$, one checks that $u \in W^{1,p}(B(0,1))$ for every $p > 1$ and that u solves (1) in the sense of distributions. One checks that for every $c > 0$, $\exp(c|Du|) \notin L^1(B(0,\frac{1}{2}))$; hence by the John–Nirenberg embedding theorem [5] (see also e.g. [11, Chapter 4, §1.3]), $Du \notin \mathrm{BMO}(B(0,\frac{1}{2}))$. □

Acknowledgments

The authors thank H. Brezis for bringing their attention on the problem. J.V.S. aknowledges the hospitality of the Mathematics Department of Rutgers University.

Références

[1] A. Ancona, Elliptic operators, conormal derivatives and positive parts of functions, with an appendix by H. Brezis, J. Funct. Anal. (2009), doi :10.1016/j.jfa.2008.12.019.

[2] H. Brezis, On a conjecture of J. Serrin, Rend. Lincei Mat. Appl. 19 (2008) 335–338.

[3] E. De Giorgi, Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari, Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) 3 (1957) 25–43.

[4] R. A. Hager, J. Ross, A regularity theorem for linear second order elliptic divergence equations, Ann. Scuola Norm. Sup. Pisa (3) 26 (1972) 283–290.

[5] F. John, L. Nirenberg, On functions of bounded mean oscillation. Comm. Pure Appl. Math. 14 (1961) 415–426.

[6] V. Kozlov, V. Maz'ya, Asymptotic formula for solutions to the Dirichlet problem for elliptic equations with discontinuous coefficients near the boundary, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 2 (2003), no. 3, 551–600.

[7] V. Kozlov, V. Maz'ya, Asymptotics of a singular solution to the Dirichlet problem for an elliptic equation with discontinuous coefficients near the boundary, in *Function spaces, differential operators and nonlinear analysis (Teistungen, 2001)*, 75–115, Birkhäuser, Basel, 2003.

[8] N. G. Meyers, An L^p-estimate for the gradient of solutions of second order elliptic divergence equations, Ann. Scuola Norm. Sup. Pisa (3) 17 (1963) 189–206.

[9] J. Serrin, Pathological solutions of elliptic differential equations. Ann. Scuola Norm. Sup. Pisa (3) 18 (1964) 385–387.

[10] E. M. Stein, Note on the class $L \log L$, Studia Math. 32 (1969) 305–310.

[11] E. M. Stein, *Harmonic analysis : real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, 43, Princeton University Press, Princeton, NJ, 1993.

Rutgers University, Department of Mathematics, 110 Frelinghuysen Road, Piscataway NJ 08854-8019, United States of America
E-mail address: kingbull@math.rutgers.edu

University of Liverpool, Department of Mathematical Sciences, Liverpool L69 3BX, United Kingdom

Linköping University, Department of Mathematics, SE-581 83 Linköping, Sweden
E-mail address: vlmaz@mai.liu.se

Université catholique de Louvain, Département de Mathématique, Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium
E-mail address: Jean.VanSchaftingen@uclouvain.be